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Logic, States, and Quantum Probabilities

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A careful analysis of mechanical descriptions provides a new logical foundation for the states and probabilities of mechanics which leads to a new understanding of quantum probabilities and their representation in Hilbert space. It is argued that all mechanical theories use a logic which is distributive but only relatively orthocomplemented, and that this, too, is the structure of its states. Probabilities are derived from this analysis using standard Kolmogorov definitions in a way that accounts for the nonstandard peculiarities of the quantum transitional probabilities as well as classicial probability assignments. At the end of the paper this analysis is used to refute arguments that quantum mechanics is nondistributive and that the failure of Bell's inequality in quantum theory threatens our conceptual scheme. Instead we reach a much less drastic interpretation of quantum mechanics.

INTRODUCTION

Probabilities are traditionally understood as assignments to "events," an understanding which arises naturally from the representation of classical mechanics in phase space. However, the development of quantum mechanics, which uses probability assignments but does not appear to have an "event space," has led to interpretive problems and also some very extreme "solutions." For example, some authors propose that we use a nondistributive logic in quantum theories, or that properties have no reality at the subatomic scale.

In this paper we reexamine the foundations, and develop a generalized analysis of probabilities which accounts for classical and quantum theories. This generalization does not violate our usual understanding of probabilities, since they remain defined in the traditional way as measures over Kolmogorov probability spaces. The key departure from tradition is in a more general sense of conditional which is introduced here. On this analysis the quantum

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transitional probabilities are strongly conditional in the sense that the probability space itself will depend on the initial condition. It is shown that classical theories do not use such strongly conditional probabilities.

Section 1 of this paper develops the logical framework for this analysis, a refinement of results first presented in Garden (1984). In Section 2 the relationship between logic, measurement, and probability assignments is made clear. A mathematical foundation for classical and quantum probabilities is presented and then applied specifically to interpreting the Hilbert space representation of quantum probabilities. Section 3 examines some arguments which are contrary to this view—specifically, we discuss arguments for the nondistributivity of quantum mechanics and the breakdown of Bell's inequality. It is argued that quantum theories are *not* nondistributive and that the failure of Bell's inequality in no way leads to a conceptual upheaval. Instead we suggest this analysis provides a simple nonradical understanding of quantum mechanics which is perhaps close in spirit to Einstein's early views.

1. PROPOSITIONAL LOGIC

1.1. The Language of Mechanics

The language used by mechanics to describe reality is derived from primitive magnitudes and their value-sets. Elementary descriptions assign values, or ranges of values, to the magnitudes and thus have a particular mathematical form—they can be represented by ordered pairs (M, Δ), where M is a magnitude of the theory T and Δ a Borel subset of values of M, i.e., $\Delta \subseteq V_M$, where $V_M \subseteq \Re$. We understand such an ordered pair as the predicate "the value of M is in Δ" or "M has a value in Δ." Such descriptions are in fact parametrized by time, as we discuss below.

The elementary language of a particular mechanical theory thus has structure generated by relations among the magnitudes, especially those derived from primitive magnitudes, and also by set relations on the value-sets of each magnitude. For example, if $p = (M, \Delta)$, then $p^\perp = (M, V_M - \Delta)$, where $V_M - \Delta$ is the set complement of Δ in V_M. Similarly, where $p_1 = (M, \Delta_1)$ and $p_2 = (M, \Delta_2)$, then $p_1 \leq p_2 \Leftrightarrow \Delta_1 \subseteq \Delta_2$ for Δ_1, Δ_2 both Borel subsets of V_M (the restriction to Borel subsets is precisely to ensure that such relations are well defined). The language appropriate to express the elementary descriptions of any theory will therefore be one which uses variables to represent the ordered pairs (M, Δ) and has at least two connectives among these variables to express the relations $\leq$ and $^\perp$ among the ordered pairs defined above. We use the following recursive definition:

Definition. The *formal language L of mechanical theory T* is generated by the following rules:

(F1) If $p = (M, \Delta)$ for M a magnitude of T and Δ a Borel subset of V_M, then $p \in L$.
(F2) If α, $\beta \in L$, then so are $\neg\alpha$ and $\alpha \supset \beta$.

The set L of well-formed formulas (wffs) thus contain variables $p, q, \ldots$ representing the elementary predicates of form (M, Δ), and complex wffs generated from these by monadic $\neg$ or binary $\supset$, which we call *negation* and *hook*, respectively. Other connectives will be introduced later by abbreviations; see below.

1.2. The Logic of Mechanical Descriptions

Truth and falsity are concepts which lie at the heart of the meaning of description, and are used to derive a logic **L** from formal language L. As usual we represent these undefined concepts by the two 'truth-values' **t** and **f**, respectively.

We leave it to metaphysics to consider the nature of *reality*, how it is that language can *describe* reality, and what therefore it "really means" to say that an elementary mechanical description is *true* or is *false* of a particular system. Answers to these metaphysical questions can sometimes distort the analysis of logic, states, and probabilities, leading to inappropriate assumptions which result in absurdity. Here we limit underlying assumptions to the minimum and we make them explicit—we assume there is a "real world" and that 'truth-values' are consistently assigned to the descriptions of mechanics in attempts to "describe" the real world. Later we assume also that measurement provides some of these truth-values. Explicitly we postulate:

(P) Mechanical theory T describes a real system by first assigning truth-values to elementary descriptions in L and then extending these in a consistent way to be truth-value assignments to complex expressions.

This assumption underlies the development of logic **L** from the language L.

First we note that strictly speaking the elementary expressions of mechanics, i.e., the predicates of form (M, Δ) are not identical with propositions. It is only when assigned a truth-value that these ordered pairs express a proposition in the sense that they express a description of reality. Also we note that in mechanical theories truth-value assignments to elementary predicates are understood as being parametrized by time, so that we as it were label the predicate with a particular time variable when we assign it a truth-value. Thus we do not simply say that "p is true of system S," but instead "p is true of S at time t_0," or alternatively, "magnitude M has a value in Δ at time t_0 on S."

Here we sometimes simplify matters by ignoring these distinctions. Thus we may occasionally ignore the difference between elementary predicates (M, Δ) and the elementary propositions expressed by (M, Δ) when truth-values are assigned. For here we shall only consider the first-order truth-functional logic derived from these elementary constituents, since this system underlies the probability assignments (although a modal extension of this logic to express probability descriptions will be sketched). In this context ambiguity will not generally result from referring to predicates as propositions. Similarly we do not always distinguish between the language L and logic of a theory, although we do use a bold **L** for the logic where this distinction is useful. And lastly in this discussion we generally ignore the time dependence of truth-value assignments. The major issues here are time-independent features of classical and quantum mechanical descriptions.

Our postulate requires consistent descriptions, and we understand this to require that truth-value assignments to wffs of a theory must respect relations among the elementary constituents—i.e., the assignments must be consistent with relations among the ordered pairs generated either by set relations on the value-sets or by relations among magnitudes established by the theory's laws. Such consistent truth-assignments we call *valuations*. A valuation h is thus a structure-preserving mapping from the wffs to the two truth-values. The structure-preserving nature of valuations will be expressed by the valuation rules, and these will also determine the nature of the logical connectives.

Note, however, that according to postulate (P) valuations need not be bivalent—they may be partial truth-value assignments taking only some members of L to $\{t, f\}$. Indeed partial assignments are most often actually used in the practice of mechanics, since these correspond to valuations where not all values of all magnitudes are known precisely, and so these partial assignments represent descriptions where probability assignments are used. Second, consistency requires that valuations preserve the relations among elementary constituents discussed above. And third, valuations must define connectives $\supset$ and $\neg$ with properties which let these connectives express implication and negation. In particular we want $\supset$ to generate a standard sense of logical equivalence. These considerations motivate the following definition, where L is the language of theory T:

Definition. A valuation h is a structure-preserving mapping $h: L \rightarrow \{t, f\}$ such that $\forall \alpha, \beta \in L$, the following hold:

(V$\neg$) If $h(\alpha) = t$, then $h(\neg\alpha) = f$.

(V$\supset$) If $h(\alpha) = t$, then $h(\alpha \supset \beta) = h(\beta)$; if $h(\alpha) = f$, then $h(\alpha \supset \beta) = t$.

If $h(\alpha) \notin \{t, f\}$ then if $h(\beta) = f$, $h(\alpha \supset \beta) \notin \{t, f\}$, and $h(\alpha \supset \beta) = t$ otherwise.

These rules preserve the structure of elementary constituents, since setting $p^{\perp} = \neg p$ ensures by ($V\neg$) that when the "the value of M is in Δ" is true, "the value of M is $V_M - \Delta$" is false. Similarly, the first part of condition ($V\supset$) preserves elementary $\leq$ by requiring that if $p \leq q$, then $h(p \supset q) = t$ for every h, so if it is true that "the value of M is in Δ," it is also true that "the value of M is in Δ^*" for $\Delta \subseteq \Delta^*$. The other parts of ($V\supset$) express properties of $\supset$ which we require to express logical deduction. There may be some debate about exactly which valuation rules are appropriate, but these conditions are guided by the requirement that $\supset$ be represented by a partial ordering, and in particular that it be reflexive, i.e., we want $(\alpha \supset \alpha)$ to be always true in order to preserve the usual sense of logical equivalence.

The definitions expressed in the valuation rules ($V\neg$) and ($V\supset$) are represented schematically in Table I, where the asterisk represents the case for any α in L, where $h(\alpha) \notin \{t, f\}$. The first columns show that the valuation rules defining $\neg$ and $\supset$ are well defined.

The following terminology is useful to a discussion of logic:

Definitions. (a) α is *true in valuation h* if $h(\alpha) = t$; α is *false in h* if $h(\alpha) = f$. (b) α is *decided in h* if $h(\alpha) \in \{t, f\}$; α is *undecided in h* otherwise. (c) α is *logically true*, $\models\alpha$, if $h(\alpha) = t$ for all valuations h of L.

We are now concerned with the structure common to the logic of any mechanical theory, i.e., the logic generated from language L and valuation

Table I. The Definitions Expressed in the Valuation Rules ($V\neg$) and ($V\supset$)[a]

α	β	$\neg\alpha$	$\alpha \supset \beta$	$\sim\alpha$	$T\alpha$	$F\alpha$	$U\alpha$	$\alpha \vee \beta$	$\alpha \wedge \beta$	$\alpha \equiv \beta$
t	t	f	t	f	t	f	f	t	t	t
t	f		f					t	f	f
t	*		*				*	t	*	*
f	t	t	t	t	f	t	f	t	f	f
f	f		t					f	f	t
f	*		t					*	f	*
*	t	*	t	t	f	f	t	t	*	*
*	f		*					*	f	*
*	*		t					*	*	t

[a]The asterisk represents the case for any α in L, where $h(\alpha) \notin \{t, f\}$. The first columns show the valuation rules defining $\neg$ and $\supset$ are well defined. The nonprimitive connectives listed here are defined by the following abbreviations: true: $T\alpha = \neg\sim\alpha$; false: $F\alpha = \neg(\neg\alpha \vee \sim\alpha)$; undecided: $U\alpha = \neg(T\alpha \vee F\alpha)$; denial: $\sim\alpha = \alpha \supset \neg\alpha$; disjunction: $\alpha \vee \beta = (\alpha \supset (\alpha \supset \beta))$; conjunction: $\alpha \wedge \beta = \neg(\neg\alpha \vee \neg\beta)$; logical equivalence: $\alpha \equiv \beta = (\alpha \supset \beta) \wedge (\beta \supset \alpha)$; as one can check.

rules ($V\neg$) and ($V\supset$) above. In fact the resulting system **L** is complete, i.e., it can be shown that the logical truths can be derived as theorems in an axiomatic system (see, e.g., Rescher, 1969, and Garden, 1984).

This logic is a generalization of classical bivalent logic in which $\supset$ is entirely classical. For example:

(Law of Identity): $\models \alpha \supset \alpha$
(Law of Distributivity): $\models (\alpha \vee (\beta \wedge \gamma)) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

Other "laws" of inference also hold in this logic. The change from classical bivalent logic comes from the fact that when bivalence is not assumed one can distinguish two different senses of "not":

(Law of Double Negation): $\models \neg\neg\alpha \equiv \alpha$
(Law of Excluded Middle): $\models \alpha \vee \sim \alpha$

We call $\neg$ *negation*, and $\sim$ *denial*. In bivalent classical logic both "laws" are satisfied by the single monadic operator. But in **L** two different monadic operators can be distinguished and each is associated with just one of these laws. Excluded Middle holds for $\sim$ but does not hold for $\neg$ in **L**.

This distinction is familiar in ordinary language and also is useful in special situations such as switching circuits or the logic of computer programs, where, for example, we need sometimes to distinguish between a switch or command having value 0 because the circuit or procedure in which it is contained has not been accessed, and having value 0 because it has been accessed but is switched "off." Similarly in ordinary language we can distinguish between "not" in the weak sense of failing to be true (denial) and in the stronger sense of the opposite being true (negation). If I deny that the pudding is sweet, for example, I may not be asserting that it is sour. Perhaps there is no pudding, or it is a painting of a pudding, or a treat for the cat, or part of a child's game with clay. . . We can readily distinguish between *negation*, "the pudding is sour," and *denial*, "it is not true that the pudding is sweet." Logic **L** therefore characterizes ordinary discourse more aptly than classical bivalent logic.

We investigate the algebraic characteristics of logic **L** by using the standard Lindenbaum–Tarski construction of a representative algebra of equivalence classes. For $\alpha, \beta \in L$, and set H of valuations of L, we have:

Definition. The (Lindenbaum–Tarski) *algebra L representing* **L** is $L = \langle [L], \leq, {}^{\perp} \rangle$, where $[L] = \{[\alpha]: \beta \in [\alpha] \text{ if } \models \alpha \equiv \beta\}$; $[\alpha] \leq [\beta]$ if $\models \alpha \supset \beta$; and $[\alpha]^{\perp} = [\neg\alpha]$.

One can show this structure is well defined in the sense that the elements of $[L]$ do not depend on the wff representing them, and the relations are similarly well defined and do not depend on the choice of wff. For example, for $\alpha, \beta \in L$, then

$$[\alpha] = [\beta] \Leftrightarrow \models \alpha \equiv \beta \Leftrightarrow \models \neg\alpha \equiv \neg\beta \Leftrightarrow [\alpha]^{\perp} = [\beta]^{\perp}$$

by the logic **L**.

Lemma 1.1. Logic **L** is represented by a distributive relatively orthocomplemented lattice *L*.

That *L* is a distributive lattice follows from the corresponding results in the logic. For example, by the definition of disjunction $[\alpha] \vee [\beta] = [\alpha \vee \beta]$, and by definition of conjunction $[\alpha \wedge \beta] = [\alpha] \wedge [\beta]$, and from the fact that distributivity holds in the logic, i.e., $\models \alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$, it follows that the lattice is distributive, i.e., $[\alpha] \vee ([\beta] \wedge [\gamma]) = ([\alpha] \vee [\beta]) \wedge ([\alpha] \vee [\gamma])$. The lattice has $1 = [(\alpha \supset \alpha)]$, $0 = [\neg(\alpha \supset \alpha)]$, but while operation $^{\perp}$ is an involution since $[\alpha]^{\perp\perp} = [\neg\neg\alpha] = [\alpha]$, it is not a full lattice complement since $[\alpha] \vee [\alpha]^{\perp} = [\alpha \vee \neg\alpha] \neq [\alpha \supset \alpha] = 1$. To see that $\neg$ is a *relative* orthocomplement we first define:

Definition. The logical *context* of α, $C_{\alpha} = \{\beta\colon \models(\alpha \wedge \neg\alpha) \supset \beta$ and $\models\beta \supset (\alpha \vee \neg\alpha)\}$.

One can show that this is a subsystem of the logic, i.e., $\langle [C_{\alpha}], \leq, ^{\perp}\rangle$ is closed with respect to the operations. By construction $[\alpha]^{\perp}$ is the complement of $[\alpha]$ relative to $[C_{\alpha}]$, for any α, as $[\alpha \wedge \neg\alpha]$ and $[\alpha \vee \neg\alpha]$ are the 0 and 1, respectively, of $[C_{\alpha}]$. Since this is also an involution as shown above, $[\alpha]^{\perp}$ is an orthocomplement of $[\alpha]$ relative to this subsystem.

The logic **L** is thus a slight generalization of classical bivalent logic where implication retains all its traditional properties but where negation has weaker properties. Algebraically the generalization from classical bivalent logic is characterized by a move from a Boolean algebra, i.e., a distributive lattice with full lattice orthocomplement, to representation by a distributive lattice which is only relatively orthocomplemented.

Note that according to this analysis the logic **L** is used by *all* mechanical theories, including classical theories. No special consideration has yet been given to quantum theories. We simply note that all mechanical theories use partial descriptions, i.e., valuations which are not bivalent, and so **L** is a more appropriate system than a bivalent logic to represent the first-order logic of mechanics.

1.3. Valuations and States

Although the *logic* used by classical and quantum theories is the same, we now examine the differences between these theories by considering the structure of *valuations* in either theory.

Definitions. For $p = (M, \Delta)$ an elementary predicate of L and $\alpha \in L$, any $h \in H$: (a) The *elementary truth-set* of h, $\mathrm{ET}_h = \{p\colon h(p) = t\}$; $\mathrm{EF}_h =$

$\{p\colon h\ (p) = f\}$; (b) The *truth-set* of h, $T_h = \{\alpha\colon h(\alpha) = t\}$; The *falsity-set* of h, $F_h = \{\beta\colon h(\beta) = f\}$.

The *elementary truth-set* of a valuation is the set of predicates true in that valuation, while the *elementary falsity-set* EF_h is the set of predicates false in h. Similarly, the *truth-set* of h is the set of all wffs true in h, and the *falsity-set* F_h of h is the set of wffs false in h. We can now define:

Definitions. (a) For h_1, h_2 in H, h_1 *contains* h_2, $h_1 \leq h_1$, if $ET_{h2} \subseteq ET_{h1}$. (b) $h^\perp \in$ H is the *converse* of h if $ET_{h\perp} = EF_h$.

One valuation *contains* another if its elementary truth-set contains the other, i.e., if it makes at least the same truth-assignments to elementary predicates. It is easy to show that if $h_2 \leq h_1$, then also $EF_{h2} \subseteq EF_{h1}$ and $T_{h2} \subseteq T_{h1}$. Clearly containment is a partial ordering on H since it is generated by set-inclusion. One valuation is the *converse* of another if all elementary predicates true in one are false in the other. The valuations of a theory are thus a system $\boldsymbol{H} = \langle H, \leq, {}^\perp\rangle$.

There is a natural correspondence between valuations in H and wffs in logic L. For any $h \in H$, $\alpha \in L$:

Definitions. (a) h_α is the *characteristic valuation* for wff α if $T_{h\alpha} = \{\beta\colon \vDash\alpha \supset \beta\}$. (b) γ_h is the *characteristic wff* for valuation h if $\vDash \gamma_h \equiv (\alpha \wedge \beta)$ for all α, $\beta \in T_h$.

The valuation h_α *characterizes wff* α in the sense that it finds true all and only the logical consequences of α. In this sense it is the "least" valuation which finds wff α true. One shows this is well defined by noting that

$$h_\alpha = h_\beta \Leftrightarrow \{\gamma\colon \vDash\alpha \supset \gamma\} = \{\gamma\colon \vDash\beta \supset \gamma\} \Leftrightarrow \vDash\alpha \equiv \beta$$

The wff γ_h *characterizes valuation* h if it is equivalent to the conjunction of all wffs found true in h. Once more one can show it is well defined since $\vDash\gamma_h \equiv \gamma_{h*} \Leftrightarrow T\gamma_h = T\gamma_{h*} \Leftrightarrow h = h^*$. There is therefore a 1–1 correspondence between (equivalence classes of) wffs and valuations:

Lemma 1.2. $[L] \approx H$.

The set of (equivalent) wffs and valuations exactly correspond. For every $\alpha \in L$ there is a corresponding $h_\alpha \in H$ by the definition, and for every $h \in H$ there is a corresponding γ_h, by the definitions above. Clearly equivalent wffs generate identical valuations by the discussion above. It is often assumed that a stronger result holds, which is that the structure of valuations $\boldsymbol{H} = \langle H, \leq, {}^\perp\rangle$ defined above is also representative of the logic, i.e., that the two structures $\boldsymbol{L}$ and $\boldsymbol{H}$ exactly coincide. Note, however, that to show that $\boldsymbol{L} = \boldsymbol{H}$ would be to show $\vDash(\alpha \supset \beta) \Leftrightarrow T_{h\alpha} \subseteq T_{h\beta}$. In fact we can show $\Rightarrow$ easily, but $\Leftarrow$ does not follow, since it may be that $T_{h\alpha} \subseteq T_{h\beta}$, yet $h(\alpha) \notin \{t, f\}$, $h(\beta) = t$, in which case $h(\alpha \supset \beta) \neq t$, so $\vDash\alpha \supset \beta$ fails.

Lemma 1.3. $\boldsymbol{H}$ is a distributive relatively orthocomplemented lattice, for any theory T.

This follows from Lemmas 1.1 and 1.2 and the discussion above: since $[\alpha] \leq [\beta] \Rightarrow h_\alpha \leq h_\beta$ for any α, $\beta \in L$, H is a distributive lattice. In fact relations $^\perp$ on either system coincide, for $[\beta] = [\alpha]^\perp \Leftrightarrow \mathrm{T}_{h\beta} = \mathrm{ET}_{h\neg\alpha} = \mathrm{EF}_{h\alpha} \Leftrightarrow h_\beta = (h_\alpha)^\perp$.

We are now ready to investigate differences between the structures $\boldsymbol{H}$ of different theories. We need some standard terminology. For any $h \in H$ the valuations of theory T:

Definitions. (a) h is *maximal* if there is no other $h^* \in H$ such that $h \subset h^*$. (b) h is *bivalent* if $\forall$ α in L, either $\alpha \in \mathrm{T}_h$ or $\neg\alpha \in \mathrm{T}_h$. (c) h is a *state* if h is maximal in H.

Maximal valuations are maximal with respect to $\leq$ and in this sense they represent the "fullest" descriptions of a theory. There are no other valuations which make the same truth-assignments to elementary predicates and also make more. *Bivalent* valuations are exhaustive with respect to $\neg$, i.e., either $h(\alpha) = t$ or $h(\neg\alpha) = t$, so either $h(\alpha) = t$ or $h(\alpha) = f$ and they are two-valued mappings from L onto the truth-set $\{t, f\}$. Clearly where they exist bivalent valuations must be maximal, but the converse is not necessarily the case, i.e., maximal valuations may not be bivalent. Lastly we call the maximal valuations *states*—these are the "fullest" valuations provided by a theory. (Later we see that "quantum state" refers to valuations which may not be states of a quantum theory; see Section 2.4 below.)

Clearly, states, the maximal valuations of a theory, may or may not be bivalent. This will depend on the structure of elementary predicates, and in particular on how the magnitudes of a theory and hence the elementary predicates are related. In what follows we are particularly interested in the difference between theories which do have bivalent valuations and those which do not. In fact we adopt the following terminology:

Definitions. (a) In a *classical theory* T every maximal valuation of L is bivalent. (b) In a *quantum theory* T maximal valuations of L are not bivalent.

We show in what follows that the condition used here to distinguish classical from quantum theories is enough to account for all the different properties of their probability assignments, thereby justifying the definition. That is, the difference between classical and quantum theories is fundamentally a difference in the structure of their valuations and hence ultimately in the structure of their elementary descriptions, i.e., their magnitudes and value-sets. The entire system of elementary predicates of a classical theory can be consistently mapped onto $\{t, f\}$, while in quantum theories only partial structure-preserving assignments of truth-values can be made.

Sometimes it is assumed that in a classical theory all *valuations* are bivalent and hence that the theory uses classical bivalent propositional logic. However, this is not supported by our actual use of classical mechanics, since in fact almost all classical descriptions of reality are imprecise and it is this very feature which makes their probability statements of interest. What characterizes classical theories is the principal expressed in the definition above—not that *every* valuation is two-valued, but rather that *every valuation of a classical theory can be extended to a two-valued one.* Quantum theories on the contrary are characterized by the fact that this is not possible, i.e., that *maximal valuations are not bivalent,* i.e., two-valued valuations do not exist.

It is important to stress that *the logic of classical and quantum descriptions is the same.* In both cases this logic is a slight generalization of classical bivalent logic where implication has all the traditional properties but where negation is weaker. One can, however, distinguish classical from quantum theories by their "*logic of states.*" In classical theories we can restrict set H to the subset H^* of bivalent states, and *this* will generate a system $\mathbf{L}^*$ which is classical bivalent logic. This follows from the fact that where the valuation rules are limited to bivalent valuations the definitions of connectives $\neg$ and $\supset$ coincide with those of bivalent classical connectives. It is well known that the representative algebra $\boldsymbol{L}^*$ of classical bivalent L^* is a Boolean algebra, i.e., distributive and fully orthocomplemented. It is also well-known by Stones' theorem that $\boldsymbol{L}^*$ is in this case equivalent to its Stone space, the system of Boolean ultrafilters containing members of L^*, and that this structure coincides with $\boldsymbol{H}^*$. So in this special case systems $\boldsymbol{H}^*$ and $\boldsymbol{L}^*$ are equivalent. In quantum theories, however, the "logic of states" will *not* be classical bivalent logic because maximal valuations are not bivalent, so although there is a subset H^* of maximal valuations in H, these do not generate bivalent logic and there is no Stone space of Boolean ultrafilters representing these states.

The special Boolean properties of $\boldsymbol{L}^*$ and $\boldsymbol{H}^*$ in classical theories have misled us about the foundation of probability assignments and especially about the role of logic and valuations in probabilities. It has been assumed that the lack in quantum theories of a bivalent Boolean "logic of states" $\boldsymbol{L}^*$ and corresponding Stone space $\boldsymbol{H}^*$ of states marks a breakdown in the logic of quantum theories and also in the probability theory, essentially because it is assumed that the structures $\boldsymbol{L}^*$ and $\boldsymbol{H}^*$ represent the logic, states, and *probability field* over which the probabilities are defined. However, we saw that the logic $\boldsymbol{L}$ and valuations $\boldsymbol{H}$ of any theory are distributive but not Boolean, and only in classical theories is there a "logic of states" $\boldsymbol{L}^*$ and system of states $\boldsymbol{H}^*$ with Boolean properties. This arises solely from the lack in quantum theories of bivalent states. However, we now argue that this does not mark a breakdown in quantum probability theory. In fact probability assignments can be defined in any mechanical theory over a standard Boolean field of sets.

2. PROBABILITIES

2.1. The Fundamental Definition

To consider how probabilities arise in mechanical theories and their relationship to logic **L**, we need to consider again the role of valuations. Recall that valuations describe reality by assigning truth-values to elementary predicates, and hence to wffs, in a structure-preserving way. The nature of reality and description are not addressed here, we simply note that reality is described when some predicates are found true, others false, of some particular real system at some particular time. But in addition to making truth-value assignments, a valuation can describe reality by assigning probabilities, telling us not that p is true or false, but there is a particular probability that p "might" be true of a system, or that proposition q is "certain to be true" of it, at a particular time. In fact in mechanical theories such predictions are most often linked to measurement on the system, so that we generally have assertions about the "likelihood" of $p = (M, \Delta)$ being found true after a measurement of the magnitude M. Measurement is discussed below in Section 2.2

First we deal with the general case of assigning a probability where no measurement is specified, and suggest the following analysis. A relation can be distinguished over H which associates with any valuation h all the other valuations which could be used after h to describe the same reality. We call this the *successor* relation and it relates to h all valuations which are *consistent* with h in the sense that they could be used in a successive description of the same reality described by h, at the same time t_0. Later we shall sketch how such an "accessibility" relation can provide a modal extension of L which does not rely on the usual "possible worlds" semantics and which expresses probabilistic descriptions.

We now formally characterize the successor relation S. There are two important conditions, reflecting that this is essentially a minimal "consistency" relation on H:

(S1) $h_2 \mathrm{S} h_1$ iff $\mathrm{ET}_{h1} \cap \mathrm{EF}_{h2} = \varnothing$.

(S2) If $h_2 \mathrm{S} h_1$, then $\forall p \in \mathrm{ET}_{h1}$, if $\exists h^* \in H$ such that $\mathrm{ET}_{h^*} = \mathrm{ET}_{h2} \cup \mathrm{T}_{hp}$, then $\mathrm{T}_{hp} \subseteq \mathrm{T}_{h2}$.

According to (S1), a successor of valuation h does not find false any elementary predicate that h finds true. This expresses our understanding that if h finds that "M has a value in Δ" is true of a system at time t_0, then a successor cannot find that "M has a value in $V_M - \Delta$" is also true of the same system at the same time t_0. According to (S2), information about a system is retained where it can be consistently combined with what was known before. Anything true in h_1 which is consistent with h_2 is also true in h_2, so that truth assignments are not simply abandoned in successive valuations if they can be consistently retained.

We now briefly sketch how relation S over H defined by (S1) and (S2) above can be used as an "accessibility" relation to define what we mean by "possible," in the sense of having "nonzero probability," or what we mean by "certain," in the sense of having "probability 1" [i.e., this relation is used to generate a modal semantics which is more general than the traditional Kripke semantics, and does not involve the notion of a "possible world"; see Garden (1984), Chapter 7]. Formally we extend logic **L** of theory T by introducing the following modal formation and valuation rules:

(F$\Box$) If $\alpha \in L$, then $\Box\alpha \in LM$.
Abbreviation: $\Diamond\alpha = df \sim \Box\alpha \sim$.
(V$\Box$) For any $h \in H$, $h(\Box\alpha) = t$ iff $\forall h^*$: $h^* \, S \, h$, $h^*(\alpha) = t$.

The formation rule (F$\Box$) introduces modal wffs of form $\Box\alpha$ into the modal extension LM, and the abbreviation adds wffs of form $\Diamond\alpha$. The valuation rule (V$\Box$) extends any valuation h of L to a corresponding valuation h of LM. According to this rule a wff is *certain* according to h, i.e., $h(\Box\alpha) = t$, if according it is true in *all* valuations which are successors to h. It is easily established that a wff is *possible* according to h [i.e., $h(\Diamond\alpha) = t$] if it is true in *some* valuation h^* which is a successor of h.

The modal extension **LM** of L, formed by adding these rules to those for **L**, allows us to express probabilistic statements about the certainty or possibility of any wff, given an initial valuation. This expresses probability statements if "certainty according to h" is understood as coinciding with "has probability 1 according to h" and "possibility according to h" coincides with "has nonzero probability assigned by h." For more discussion of this modal extension see Garden (1984). Here we move directly to develop these ideas for the logical foundation of probability assignments in mechanics. We note that a natural way of developing these ideas is to suggest that the *probability assigned by h to p to describe the system at time t_0* will be a *measure of the successor valuations of h which find p true of the system at* t_0. In what follows we first formalize this as a definition, then show in the rest of the paper how very successful this is as a generalized foundation for probabilities, yielding classical assignments in classical theories, but strong conditional probabilities in nonclassical theories which have all the key peculiarities of quantum transitional probabilities. For the first time we can understand the quantum assignments as generalized but essentially classical probabilities.

We now seek to formalize the analysis sketched above. Probabilities assigned by h are to be measures of appropriate successor-sets of h. In formalizing this insight we first note that in taking a "measure" of the successor valuations of h we can restrict attention to the *maximal* valuations. Every valuation h is contained in a maximal valuation, by definition of maximal, and so in "counting" the successors we can "count" just the maximal successors. We now define the following sets. For any $h \in H$, $\alpha \in L$:

Definitions. (a) The *successor-set* $S_h = \{h^*: h^* \in H, h^*$ is maximal, and $h^* Sh\}$. (b) The α-*successor-set* $S_h^\alpha = \{h^*: h^* \in S_h$ and $h(\alpha) = t\}$.

The successor-set contains all *states* which are successors of h, i.e., S_h contains all maximal valuations meeting conditions (S1) and (S2) above. The α-successor set contains those states in S_h which find wff α true. These sets will provide the logical foundation for probability statements. We say that wff α has nonzero probability according to h if $S_h^\alpha \neq \emptyset$, i.e., if α is true in some successor valuation of h. Similarly wff α has probability 1 according to h if $S_h^\alpha = S_h$, i.e., if all successor states of h find α true. Extending this analysis, we say that *the probability of* α *according to* h will be a *measure of how near the* α*-successor-set of* h *is to being the whole successor-set of* h.

A proper mathematical foundation for this analysis is provided when we identify a *probability space* over which these measures are defined. According to standard Kolmogorov probability theory this must be a triple $(X, \mathfrak{I}, \mu)$, where **X** is a set, $\mathfrak{I}$ a field of subsets of X, and μ a probability measure defined over this field. In fact the definitions above provides such a space. For any h we have the set S_h of successor-states, and from this we can generate a field of subsets using the definition above of α-successor-sets. This provides the fundamental definition first presented in Garden (1984):

Fundamental Definition. For any $h \in H$ there is *a Kolmogorov probability space* $\langle S_h, \mathfrak{I}_h, \mu_h\rangle$, where S_h is the successor-set of h and $\mathfrak{I}_h = \langle F_h, \cap, \cup, -\rangle$ is the field of subsets of S_h where $F_h = \{S_h^\alpha: \alpha \in L\}$, and μ_h is a probability assignment over $\mathfrak{I}_h$.

The *probability of* α *according to initial condition* h, $\text{prob}_h(\alpha) = \mu_h(S_h^\alpha)$.

The probability space for any h is generated by its successor-states S_h and a field of subsets of this set. We establish that $\mathfrak{I}_h$ is indeed a field of subsets of S_h by showing that set operations correspond to logical connectives. For example, $h^* \in S_h^\alpha \cup S_h^\beta \Rightarrow h^*$ S h and either $h^*(\alpha) = t$ or $h^*(\beta) = t \Rightarrow h^* \in S_h^{\alpha\vee\beta}$, and similarly $h^* \in S_h^{\alpha\vee\beta} \Rightarrow h^*$ S h and $h^*(\alpha \vee \beta) = t \Rightarrow$

$h^* \in S_h^\alpha \cup S_h^\beta$, and so $S_h^\alpha \cup S_h^\beta = S_h^{\alpha\vee\beta}$. If countable disjunctions are allowed in L, then $\Im_h$ is a σ-field. Note that it is connective $\sim$ which generates set complement among the subsets. For where $-$ is set complementation in $\Im_h$,

$$h^* \in -(S_h^\alpha) \Leftrightarrow h^* (\alpha) \neq t \;\&\; h^* \mathrm{S}\, h \Leftrightarrow h^* (\sim\alpha) = t \;\&\; h^* \mathrm{S}\, h \Leftrightarrow h^* \in S_h^{\sim\alpha}$$

and so $-(S_h^\alpha) = S_h^{\sim\alpha}$. The probability space defined is thus a standard Kolmogorov space and μ_h is a standard probability measure. The last part of the definition tells us that the probability of α, given initial h, is a probability measure over this field of the α-successor-set of h.

From the natural correspondence between wffs in L and valuations in H we derive the expressions:

Definitions. (a) $\mathrm{prob}_\alpha(\beta) = \mathrm{prob}_{h\alpha}(\beta)$, where h_α is the characteristic valuation for α; (b) $\mathrm{prob}_h(h^*) = \mathrm{prob}_h(\gamma^*)$, where γ^* is the characteristic wff for h^*.

That is, probabilities may be assigned from wffs to wffs, or from valuations to valuations.

The fundamental definition makes clear that the *probabilities are standard Kolmogorov probability measures* and therefore have all the traditional properties associated with probability assignments. Probabilities are measures over a Boolean field and the set operations on this field are reflected in the probability assignments in the usual way. So this analysis does *not* involve generalizing our usual notion of a probability in the sense of changing the properties of the measure or the properties of the underlying set structure which is so important to the definition. This is a generalization of traditional ideas only in one respect, and that is in regarding the probabilities as *fundamentally conditional on the initial condition.* The probabilities are *strongly conditional* in the sense that the choice of probability space itself depends on the initial condition.

This fundamental definition also makes clear the relationship between *logic, valuations*, and *probability assignments*. Neither the logic $\boldsymbol{L}$ nor the system $\boldsymbol{H}$ of valuations of a theory provides the field of sets in the probability space for the theory's probability assignments, and neither does the "logic of states" $\mathbf{L}^*$ or the system of states H^*. According to the fundamental definition above, the space uses $\Im_h$, which is a Boolean field of subsets generated from a subset of the states determined by initial condition h.

2.2. Classical Probabilities

We now show that the familiar probability assignments of classical mechanics can be derived from the fundamental definition as a special case. Classical theories do not use the strongly conditional probabilities, as we now proceed to show. However, before demonstrating this it is useful to introduce the traditional sense of "conditional" probability:

$$\textit{Definition}: \text{prob}_h(\alpha \mid \beta) = \begin{cases} \text{prob}_h(\alpha \wedge \beta) \mid \text{prob}_h(\beta) & \text{where } \text{prob}_h(\beta) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We call this the *weak* conditional (or "ratio" probability) to distinguish it from the *strong* dependence in the fundamental definition on initial condition h. It follows from the definition that $\text{prob}_h(\alpha \mid \beta) = \mu_h(S_h^\alpha \cup S_h^\beta) \mid \mu_h(S_h^\beta)$ if $S_h^\beta \neq \varnothing$, and $= 0$ otherwise, and so this does correspond to the traditional notion of conditional probability used in standard probability theory.

We can also derive an "unconditional" probability assignment as a special case of the fundamental definition. In any theory T, logic $\mathbf{L}$, $\alpha \in L$:

Definitions. (a) The *trivial valuation* h_0 is such that $T_{h0} = \{\alpha: \vDash\alpha\}$. (b) The unconditional probability of α, $prob(\alpha) = \text{prob}_{h0}(\alpha)$.

The trivial valuation finds only logical truths of $\mathbf{L}$ to be true and the "unconditional" probability is conditional on this valuation.

We are now ready to consider the special properties of classical probabilities. First we show for any set H of valuations:

Lemma 2.1. In a classical theory S $= \leq$. *In a quantum theory* S $\neq \leq$.

Recall that in a classical theory all maximal valuations are bivalent, and so in a classical theory where h_2 S h_1 condition (S1) on S ensures that $\exists h_3$ such that $\text{ET}_{h3} = \text{ET}_{h1} \cup \text{ET}_{h2}$ by the existence of bivalent maximal valuations, and so by (S2) $h_3 = h_2$ and $h_1 \leq h_2$. Thus in a classical theory h_2 S $h_1 \Leftrightarrow h_1 \leq h_2 \Leftrightarrow \text{ET}_{h_1} \subseteq \text{ET}_{h_2}$. However, in the general quantum case states are not bivalent and so we cannot guarantee the existence of such an h_3 and so S does not coincide with $\leq$.

Lemma 2.1 establishes the very special property of classical theories, that we can regard successive descriptions of the same reality as "building up" truth-sets until the final two-valued state is reached, uniquely characterizing a reality within the theory. In classical theories we can assume that successive descriptions of the same reality at the same time t_0 simply retain all truth-values until a unique bivalent state is achieved to characterize this reality. However, in quantum theories we do not have bivalent valuations and so in these theories succession and containment do not coincide. This means we cannot assume in quantum theories that all truth-assignments are retained in subsequent descriptions of the same reality at the same time t_0, and so successive valuations do not simply "build up" truth-values. Two different states may be used to describe the same unchanged reality at time t_0. A wff (including an elementary predicate) which is true in one may be left undecided in the other, and the unique correspondence between realities and maximal valuations is lost in quantum theories.

An important consequence of Lemma 2.1 is the following:

Lemma 2.2. For any $h \in \mathrm{H}$, $\alpha \in L$, γ_h the characteristic wff for h: *In a classical theory* $\mathrm{prob}_h(\alpha) = \mathrm{prob}(\alpha|\gamma_h)$. *In a quantum theory* $\mathrm{prob}_h(\alpha) \neq \mathrm{prob}(\alpha|\gamma_h)$.

In classical theories the strong and weak conditional probabilities coincide, in nonclassical theories they do not. For by Lemma 2.1, in a classical theory $S_h^\alpha = \{h^*\colon h^* \mathrm{S}\, h \text{ and } h^*(\alpha) = t\} = \{h^*\colon Th \subseteq Th^* \;\&\; h^*(\alpha) = t\} = S^{\alpha\wedge\gamma h}$ and so $\mathrm{prob}_h(\alpha) = \mu_h(S_h^\alpha) = \mu(S^{\alpha\wedge\gamma h})|\mu(S^{\gamma h}) = \mathrm{prob}(\alpha|\gamma_h)$. In quantum theories, however, where $S \neq \leq$, $\mathrm{ET}_h \not\subset \mathrm{ET}_{h^*}$ when $h^* \mathrm{S}\, h$, and so $S_h^\alpha \neq S^{\alpha\wedge\gamma h}$ and hence $\mathrm{prob}_h(\alpha) \neq \mathrm{prob}(\alpha|\gamma_h)$.

It follows from Lemma 2.2 that *in the special case of a classical theory only a single probability space is needed to generate all probability assignments of the theory.* This is the "trivial" probability space $(S_0, \Im_0, \mu_0) = (H^*, \Im_0, \mu_0)$, i.e., it is derived from H^*, the set of all states of the theory. However, although this space can be *defined* in any theory and used to generate the "unconditional" probabilities of any theory, in nonclassical theories it simply does not have the role of a fundamental "event space" from which all of the other fundamentally conditional probabilities can be derived. It has this special property only in classical theories, by Lemma 2.2. In quantum theories probabilities conditional on nontrival *h cannot* be expressed as standard weak conditional probabilities on $(H^*, \Im_0, \mu_0)$, the "unconditional" space of the theory. The strong conditionals do not coincide with weak conditionals, and a single probability space cannot generate all conditional probability assignments of the theory.

It follows from the fundamental definition that in classical theories $\Im_0$ is just the Stone space $\boldsymbol{H^*}$ of the "logic of states" $\boldsymbol{L^*}$ discussed above. So in this special classical case $\boldsymbol{L^*}$, $\boldsymbol{H^*}$, and $\Im_0$ are all equivalent. This had led to the assumption that the logic, states, and the probability field of a theory coincide, and has made the lack of a single "event space" in quantum mechanics particularly mysterious. However, here we see the mystery resolved. The "logic of states" $\boldsymbol{L^*}$ and the system of maximal valuations $\boldsymbol{H^*}$ are *not* fundamental to probability structures. Instead they happen to coincide with field $\Im_0$ only in the special case of theories with bivalent states. In a theory where states are not bivalent, $\boldsymbol{L^*}$ and $\boldsymbol{H^*}$ are not Boolean, they do not therefore represent a probability "field," and there is in fact no single space to generate all the probabilities of a theory. Instead these probabilities will be strongly conditional, defined over a probability space which is generated from a subset of states depending on the initial condition.

2.3. Measurement

Probability assignments in mechanical theories are most often concerned with the outcome of measurement and we now show how measurement can be logically analyzed and formally expressed. The physical and metaphysical nature of measurement will not be considered, i.e., we do not attempt to answer the questions of what the metaphysical or physical nature of measurement is, of how and why it helps us to describe reality. We consider here only its logical characteristics, accepting that measurement is some process, in principle definable, which provides truth-values of elementary predicates in order to describe a real system at a particular time. In fact measurement *of a particular magnitude* M results in the assignment of truth-values to elementary M-predicates, and so we use M for a magnitude or a measurement of the magnitude without ambiguity. The key logical characteristic of a measurement M is that it results in truth-value assignments to M-predicates in the theory.

Note particularly that no *physical* assumptions are made about the measurement process and in particular there is no assumption about disturbance of a system during measurement nor even about time advancing during the measuring process. We assume here that measurement is "ideal" in the sense that it really has no physical properties at all. It may be a fact of the world that all measurement disturbs and that all measurement takes time, but these aspects are ignored here because we are concerned solely with the logical characteristics of measurement, to see what follows from this logical characterization alone.

We can now sum up the logical properties of measurement in the following way. In any theory T:

Definition. After a *measurement of magnitude* M, a system is described by $h_M \in H$ such that:

(M1) $h_M(p) = t$ for some M-predicate p, i.e., some $p = (M, \Delta)$, $p \in L$.
(M2) h_M S h.

Condition (M1) expresses the requirement that a measurement of magnitude M finds an M-predicate true, i.e., it designates a value or range of values as true of magnitude M on the system measured. Condition (M2) expresses the assumption that the measurement is ideal in the sense that the system described before and after the measurement is assumed to be the same real system at the same time t_0.

We are now interested in the properties of measurement and in particular properties of valuation h_M which can be derived from this logical characterization. First we need the terminology:

Definitions. (a) The *trivial M-predicate* is $1_M = (M, V_M)$. (b) The *outcome of a particular measurement of M* will be some $p = (M, \Delta)$ such that $h_M(p) = t$ and there is no $q = (M, \Delta^*)$ where $h(q) = t$ and $\Delta^* \subset \Delta$.

Predicate 1_M assigns the whole value-set V_M to M. It is *trivial* for M in the sense that it will be true when any other M-predicate is true and hence by (M1) will be true after any measurement of M. For we know for any $p = (M, \Delta)$, since $\Delta \subseteq V_M$, $\models (p \supset 1_M)$ by definition of valuation. The *outcome* of a measurement M is the "least" M-predicate which is found true after a measurement of M. Note we do not require measurement to be "ideally precise" since we do not require that measurement must result in assignment of truth to an atomic proposition (M, r) for real number r in V_M. Instead any M-predicate may be an outcome of M.

From the logical properties above one can now conclude that:

Lemma 2.3. In a quantum theory the state describing a system will change after measurement; in a classical theory this is never the case.

This result follows simply from conditions (M1) and (M2) and Lemma 2.1. Where succession is containment and states are bivalent, one can assume that the valuation h_M which describes the system after measurement, and the initial condition h, are both contained in the same bivalent state and hence one assumes that although the *valuation* changes after measurement, the underlying *state* is unchanged. In this sense measurement in classical theories can always be assumed "non-disturbing" to earlier truth-value assignments. However, in a quantum theory the situation is quite different. Where maximal valuations are not bivalent there is in general no state h^* which contains h and h_M, and hence the state h_1 describing the system before measurement, which contains h, may be different from the state h_2 describing the system after measurement, which contains h_M. This result stems solely from the logical properties of measurement together with the lack of bivalent valuations. Where there are no bivalent valuations, measurements lead to drastic but predictable changes in state reflecting the fact that new information is introduced which may not be combined consistently with existing truth-values. The changes are predictable in the sense that they depend on M, the magnitude measured.

Probabilities conditional on measurement are in fact weakly conditional on the trivial *M-predicate*. That is, for any theory T, $h \in H$, $\alpha \in L$:

Lemma 2.4. The probability that α is true if a system initially described by h is measured for M is given by $\text{prob}_h(\alpha{:}M) = \text{prob}_h(\alpha \mid 1_M)$.

The probability of finding α true after a measurement of M on a system initially described by h is the strongly conditional probability of α given initial h, (weakly) conditional on the trivial M-predicate being true. For it follows from (M1) that some M-predicate must be true after M, so

$$\text{prob}_h(\alpha{:}M) = \mu_h(S_h^{\alpha\wedge p})|\mu_h(S_h^p) = \text{prob}_h(\alpha|p)$$

for arbitrary M-outcome p. But as we saw above, for any $p = (M, \Delta)$, $\Delta \subseteq V_M$ and so

$$\text{prob}_h(\alpha{:}M) = \mu_h(S_h^{\alpha\wedge 1M})|\mu_h(S_h^{1M}) = \text{prob}_h(\alpha|1_M)$$

Only those valuations which assign truth to an M-predicate, and hence find 1_M true, are "counted" in the probability assigned to any wff after magnitude M is measured.

We pause briefly to consider how measurement-specific probability assignments can be expressed in a modal extension of L. A magnitude-dependent modal operator $\Box_M$ can be defined with a formation rule analogous to that already given for $\Box$ (in Section 2.1 above), and the following valuation rule:

$$(\text{V}\Box_M)\quad h(\Box_M\alpha) = t \text{ iff } h^*(\alpha) = t\ \forall h^*\text{: } h^*\ \text{S}\ h \text{ and } h^*(1_M) = t.$$

Wff α *is certain after measurement of M according to h*, if α is true in all valuations accessible from h which decide the trivial M-predicate, or equivalently which find some M-outcome true. Clearly in general $\Box_M$ does not coincide with operator $\Box$ since in general there will be states that are successors of h which do not decide 1_M, for in general nonclassical states are not bivalent.

Lemma 2.4 can be used immediately to establish that in classical theories some very special results obtain for measurements of two different magnitudes M, N performed in turn on a system initially described by h. We let prob$_h(\alpha{:}M)(\beta{:}N)$ be the probability of first finding α true after a measurement of M on a system initially described by h, then finding β true after a subsequent measurement of N on the same system (still at the same time t_0). Then:

Lemma 2.5. $\forall h \in H$, α, $\beta \in L$, magnitudes M, N: *In a classical theory*

$$\text{prob}_h(\alpha{:}M)(\beta{:}N) = \text{prob}_h(\beta{:}N)(\alpha{:}M)$$

In a quantum theory

$$\text{prob}_h(\alpha{:}M)(\beta{:}N) \neq \text{prob}_h(\beta{:}N)(\alpha{:}M)$$

According to this result, the order of measurements makes no difference to sequential probabilities in classical theories, but in quantum theories the probability of a sequence will depend on the order. To simplify the terminology we set $S_M = S_{hM}$ and $S^M = S^{1M}$. Then in general $\text{prob}_h(\alpha{:}M)(\beta{:}N) = \mu_{h^*}(S_N^\beta)$, where $h^* \in S_M^\alpha$, while $\text{prob}_h(\beta{:}N)(\alpha{:}M) = \mu_{h_\#}(S_M^\alpha)$ where $h_\# \in S_N^\beta$ and these measures and sets are simply not in general the same. Only in classical theories by the lemmas above do we get the result that for arbitrary α, M, $S_M^\alpha = S^{M\wedge\alpha}$ and so in this special case

$$\text{prob}_h(\alpha{:}M)(\beta{:}N) = \mu(S^{\alpha\wedge M\wedge\beta\wedge\gamma^h})|\mu(S^{\gamma^h}) = \text{prob}_h(\beta{:}N)(\alpha{:}M)$$

i.e., the probability of a sequence coincides with the probability of a corresponding conjunction, and since the order of conjuncts is irrelevant in the logic **L** the order of items in a sequence is irrelevant to the sequential probability in a classical theory. However, in the general case, where probabilities are strongly conditional, sequences cannot be reduced to conjunctions and the order of items in the sequence will be crucial to the probability.

So this analysis does yield features of the quantum transitional probabilities as well as the familiar classical conditionals. States change after measurement in a drastic but well-defined way which depends on the magnitude measured (Lemma 2.3). And probabilities in quantum theories depend on the order in which magnitudes are measured in a sequence (Lemma 2.5). We have seen that these results are consequences solely of logical properties of measurement. Since classical and quantum theories differ only in the structure of their valuations, this difference alone must account for these very different properties of measurement in classical and quantum mechanics.

2.4. Phase Space and Hilbert Space

We now relate this logical analysis to the formalism actually used in mechanics, i.e., to the classical phase space and to the Hilbert space representation of quantum theories.

Classical phase space is a space of "points," each corresponding to precise values for each primitive magnitude in a theory. For example, a classical theory describing a system with just one degree of freedom has a phase space of ordered pairs (q_n, p_k), where the q_n, p_k are Q- and P-values, respectively. Strictly speaking these "points" in phase space are parametrized by time, so that the system at time t_0 will be described by an ordered pair of position and momentum values $(q(t_0), p(t_0))$. In general a system with n degrees of freedom is represented by a $2n$-dimensional phase space of points of form $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$. The probabilities of classical mechanics are probability measures over the field of subsets of such a phase space. The magnitudes (i.e., "observables") of the theory are random variables over the phase space mapping the "points" of phase space to real numbers. For example, magnitude Q_i (representing position in the ith direction) is a function taking point $(q_1(t), \ldots, q_n(t), p_1(t), \ldots, p_n(t))$ to its ith component $q_i(t)$.

It is easy to see that the classical phase space representation of classical mechanics can be derived from the logical analysis above. A classical theory has all maximal valuations bivalent, so a classical state h corresponds to a truth-value assignment to all elementary predicates including all *atomic* predicates of form (M, r) for r in V_M. Thus h corresponds to the "point" $(r_1, r_2, \ldots, r_n)$, where the r_i are appropriate M_i-values, $r_i \in V_{Mi}$. So the set of all classical states H^* and the set of points in phase space exactly correspond, as do the field of subsets generated from these sets. In a similar way we can also represent magnitudes as random variables over the bivalent valuations so that M takes state h to the M-value in the atomic *M-predicate* which is true in h, i.e., M: $H \to \Re$ defined by $M(h) = r_i$, for $r_i \in V_M$ such that $h(M, r_i) = t$, for h a state. All features of the classical phase space can be derived in a similar way from the logical analysis where a theory is classical.

It is well known that the phase space representation of classical mechanics is impossible in quantum theory. It is now accepted that theories of quantum mechanics are represented using the formalism of separable complete Hilbert space. According to this representation the "pure states" of the quantum theory correspond to normed unit vectors in a complete separable Hilbert space associated with the theory. Quantum "mixtures" are probability distributions over the pure states. The magnitudes of a quantum theory are represented by special (linear Hermitian) "observable" operators on the Hilbert space. The special properties of these operators associate them with a particular set of "axes" or an analogous set of special subspaces in the space, each corresponding to an element in the spectrum of A, i.e., to a member of its value-set.

The "decomposition" of any vector ϕ in the Hilbert space in terms of its components along the axes or in the subspaces of the observable operator generates the probability assignments according to initial state ϕ of the corresponding outcomes of measuring the magnitude. In the simplest case of operator A with discrete distinct eigenvalues, A is associated with eigenbasis $\{\alpha_i\}$ and corresponding eigenvalues a_i, i.e., $A = \Sigma_i a_i \alpha_i$, and any vector ϕ in the space can be expressed in terms of components along the eigenvectors in this eigenbasis, i.e., $\phi = \Sigma_i(\alpha_i, \phi)\alpha_i$ for any vector ϕ. It is the "size" of the component of ϕ along α_i, $|(\alpha_i, \phi)|^2$, which gives the probability that after a measurement of A the value of A will be a_i, i.e., that the system will be in state α_i.

This analysis can also be stated in terms of projection operators. Each subspace of Hilbert space is associated with a corresponding projection operator, and in the simple case above the "decomposition" of vector ϕ is reexpressed by noting that $(\alpha_i, \phi)\alpha_i = P_{\alpha i}\phi$, i.e., the component of ϕ along α_i

is the projection of ϕ onto the one-dimensional subspace α_i, so that the probability of finding value a_i given initial state vector ϕ is $|(\alpha_i, \phi)|^2 = \|P_{ai}\phi\|^2$. This reexpression is useful for the general case where A may have continuous or degenerate spectrum. The Spectral Theorem lets us express any observable operator A in terms of the decomposition $A = \int_{\infty}^{+\infty} \lambda \, dE(\lambda)$, where the $E(\lambda)$ are projection operators with properties analogous to those of a basis set and the λ are contained in the spectrum of A. In this case the probability that the value of A will be found in the interval $\lambda_2 - \lambda_1$ is $\text{prob}_\phi(\lambda_2 - \lambda_1) = \|E(\lambda_2) - E(\lambda_1)\phi\|^2$, i.e., as with the simple case, the probability that A has a value in interval $(\lambda_2 - \lambda_1)$ of its spectrum is given by the "size" of the projection of ϕ into the subspace associated with this interval.

To understand the implications of this representation we need only appreciate the case of simple A with discrete eigenvalues a_i and corresponding eigenvectors α_i. As discussed above, quantum theory tells us that the probability of finding value a_i after a measurement of A on a system initially described by ϕ is $\text{prob}_\phi(a_i{:}A) = |(\phi, \alpha_i)|^2 = \|P_{ai}\phi\|^2$. According to Born's interpretation of the quantum state, *this is the most that* ϕ *can tell us about magnitude* A. It follows for example that value a_i can be predicted with certainty only in the case where ϕ coincides with α_i, since only in this case will $\text{prob}_\phi(a_i{:}A) = |(\phi, \phi)|^2 = 1$. Yet even this state will only make statistical predictions about the outcome of the measurement of other observables, for in general $\text{prob}_{\alpha i}(b_k{:}B) \neq 1$ where $B \neq A$. Quantum states are irreducibly statistical.

Although this is a simplification of the mathematics, we see here the heart of the problem of understanding quantum probabilities. Classical phase space gives way to a vector space representation where probability assignments are generated by the inner product, the metric on the space, so that the "size" of projections of the vector into particular subspaces associated with the observable operator gives the probabilities that corresponding values of the observable hold. This mathematical formalism is intractable to representation in terms of a single "phase space" and hence has seemed intractable to analysis in terms of traditional probability measures. However, the present logical analysis provides exactly this understanding of quantum probabilities.

First note that the quantum *states* ϕ, i.e., the pure states or mixtures represented by vectors in Hilbert space, correspond to (not necessarily maximal) *valuations h* of a quantum theory in the logical analysis. According to this analysis there is a correspondence between the Hilbert space representing quantum theory T and the system $\boldsymbol{H} = \langle H, \leq, {}^\perp \rangle$ of valuations of T. The "pure" states of the theory correspond to valuations which are characteristic of the atomic predicates. For example, the "pure" state α_i, an eigenvector of simple observable operator A corresponding to eigenvalue a_i, corresponds to valuation h_p characteristic for $p = (A, a_i)$. Quantum "mixtures" are probability assignments to the pure states and these, too, will correspond to valuations of the theory, but to valuations which are not characteristic of atomic predicates.

We have seen that probabilities are generated in Hilbert space by the projection of the initial state vector onto special "axes" or subspaces associated with observable operators. In the simple case this is the projection onto the eigenvectors of the observable operator. According to the logical analysis these projections, i.e., the size of components of ϕ along the axes of A, represent the strongly conditional probabilities of the fundamental definition. The probability $\text{prob}_\phi(a_i{:}A) = |(\phi, \alpha_i)|^2 = \|P_{\alpha i}\phi\|^2$ in Hilbert space corresponds to probability $\text{prob}_\phi(p{:}A)$ for $p = (A, a_i)$ i.e., to the probability $\text{prob}_\phi(h_p{:}A)$ in the subspaces of $\boldsymbol{H}$. According to this correspondence, the Hilbert space projections in the decomposition of ϕ in terms of A, which give a measure of the "overlap" of ϕ with each of the eigenvectors, correspond in the logical analysis to measures of the "overlap" in the sense of "consistency" between initial valuation ϕ and the characteristic atomic valuations associated with A, that is, to the proportion of successor states of ϕ deciding 1_A which contain these atomic valuations. There is therefore an analogy between the "size" of projections into subspaces of Hilbert space and the measures of subspaces provided by the fundamental definition over $\boldsymbol{H}$.

We now check that in some simple cases the logical analysis does indeed concur with the Hilbert space representation. For example, suppose ϕ is actually an eigenvector α_i of simple operator A; then quantum mechanics tells us that in this case $\text{prob}_\phi(a_i{:}A) = |(\phi, \phi)|^2 = \|P_{\alpha i}\alpha_i\|^2 = 1$. Furthermore, quantum mechanics tells us that if a_k is some other value of A, i.e., $i \neq k$, then since in this case $(\alpha_k, \alpha_i) = 0$, also $\text{prob}_\phi(a_k{:}A) = 0$. We now show that similar results hold according to the logical analysis.

Lemma 2.6. If $p = (A, a_i)$, $q = (A, a_k)$, for $i \neq k$, and $p, q \in L$, then

$$\text{prob}_p(p{:}A) = 1 \qquad \text{and} \qquad \text{prob}_p(q{:}A) = 0$$

If the initial valuation describing a system is characteristic of an atomic predicate, then there is probability 1 that a measurement of this magnitude will result in this atomic value being found true of this system, and there is probability 0 that some other value of the magnitude is found true after the measurement. Here $\text{prob}_p = \text{prob}_{hp}$, i.e., by assumption of this special case the initial condition is the characteristic valuation h_p where $p = (A, a_i)$. According to the fundamental definition and the terminology introduced in Section 2.1, $\text{prob}_p(p{:}A) = \mu_p(S_p^{p\wedge A})|\mu_p(S_p^A)$ (where for simplicity we let $1_A = A$), by Lemma 2.3. Then since $\models(p \supset 1_A)$ for any A-outcome p by definition of valuation, by condition (S2), $S_p^{p\wedge A}$ for arbitrary p is equal to S_p^A and so $\text{prob}_p(p{:}A) = 1$. Similarly, if $q = (a_k, A)$, $k \neq i$, then $\text{prob}_p(q{:}A) = \mu_p(S_p^{q\wedge A})|\mu_p(S_p^A)$, but since $h_p(p) = t$, $h_p(q) = f$ by definition of valuation, and so by (S1) there is no h^* S h_p such that h^* $(q) = t$, and so $S_p^{q \wedge A} = \varnothing$, and $\text{prob}_p(q{:}A) = 0$. Thus the logical analysis agrees with the quantum results in these cases.

It also follows from Lemma 2.6 that in the case of classical theories, where all maximal valuations are bivalent, every probability assigned by a *state* of the theory will be either 1 or 0, i.e., classical states are dispersion free.

We now end this section by giving a blow-by-blow account of how each of the key peculiarities of the quantum transitional probabilities which arise from their mathematical representation in Hilbert space is also a peculiarity of the strongly conditional probabilities and can therefore be explained on this analysis. The numbered points are taken by some authors to be postulates, axiomatic to acceptance of quantum mechanics. Certainly all are fundamental and are usually understood as representing the intractable, irreducibly "non-classical" features of quantum theory. We now show how each is explained according to this logical analysis.

1. *Vectors in Hilbert space represent the quantum states:* This was the subject of discussion above. We have seen that classical theories use a single probability space to generate all their probabilities, but in quantum theories, where maximal valuations are not bivalent, a phase space representation is not possible, as the single "unconditional" probability space cannot generate all the (strongly) conditional probabilities of the theory. According to the logical analysis the probabilities of a quantum theory are generated from the Kolmogorov probability spaces of the fundamental definition, and hence the theory will be represented by the system $\boldsymbol{H} = \langle H, \leq, {}^{\perp}\rangle$ of valuations over which these probability spaces are defined. Already we have seen some striking similarities between this analysis and the Hilbert space representation of quantum mechanics—predictions vary with the magnitude measured (Lemma 2.3), probabilities depend on the order of magnitudes in a sequence of measurements (Lemma 2.5), and in some special cases the probabilities coincide (Lemma 2.6).

There are in fact some natural analogies between the structure of $\boldsymbol{H}$ and Hilbert space. Already we suggested that the vectors representing "quantum states" in Hilbert space correspond to (not necessarily maximal) valuations of a theory—the unit "rays" in Hilbert space corresponding to valuations characteristic of the atomic predicates of the theory, and hence to these atomic predicates themselves. The fundamental definition of probability (Section 2.1) also provides a kind of "metric" on $\boldsymbol{H}$, since it associates with valuations (h_1, h_2) a real number $\text{prob}_{h_1}(h_2)$, which is a measure of the "overlap" or consistency of the two valuations—a measure of the extent to which the successors of the first valuation agree with truth-assignments made by the second. This is in obvious analogy to the Hilbert space inner product and especially its expression in terms of projections in the subspaces of Hilbert space.

A full and accurate analysis of Hilbert space, and of system $\boldsymbol{H}$, is needed to demonstrate that all features of the Hilbert space representation have a counterpart in the logical analysis. Here we simply indicate the main features of this analysis in the points below. In Section 3 we analyze the correspondence between "properties" and subspaces of Hilbert space, arguing that contrary to accepted opinion, quantum properties are *not* nondistributive, a conclusion which also supports the view that $\boldsymbol{H} = \langle H, \leq, {}^{\perp}\rangle$ essentially corresponds to the Hilbert space representing quantum mechanics. At the very least this discussion shows that clear analogies exist between $\boldsymbol{H}$ and Hilbert space, and this shows how a structure such as Hilbert space which is not Boolean can represent the predictions of mechanics.

2. *Observables as linear Hermitian operators on the Hilbert space:* According to quantum theory, magnitudes are represented by observable operators which "pick out" special subspaces of states (an eigenbasis of vectors in the simple case), so that any other vector can be broken down into components in these subspaces and the "size" of these components determines the probability that the corresponding value or interval in the spectrum obtains according to this initial state. The logical analysis is analogous to this, since here, too, each magnitude of theory T corresponds naturally with the valuations that are characteristic for M-outcomes. In the simple case of magnitude M with discrete distinct M-values, M "picks out" the ("eigenbasis" of) valuations characteristic for its atomic predicates, i.e., set $\{h_i\}$, where each h_i is characteristic for $p_i = (M, r_i)$, $r_i \in V_M$. Furthermore, the probability assigned by initial ϕ to value r_i of M will according to the fundamental definition be a measure of the proportion of successor states of ϕ deciding M which contain h_i, an analogy to the Hilbert space representation according to which the probability is given by the "size" of the projection of ϕ along this eigenvector. In the case where a magnitude is not simple, i.e., the spectrum is degenerate or continuous, this logical analysis can be restated in a way that is exactly analogous to restatement in the Hilbert space representation, since in this case a family of "subspaces" in H can be "picked out" corresponding to valuations characteristic of nonatomic elementary M-predicates.

The Hilbert space analysis of magnitudes as operators on vectors can also be restated in the logical terms, for we can define "operator M" over H as taking valuation h to another valuation h_p which is characteristic for the "least" M-predicate consistent with h, i.e., M maps h to h_p for $p \in L$, $h^*(p) = t$ for h^* in S_h, and $p = (M, \Delta)$, where there is no $\Delta' \subset \Delta$ such that $p' = (M, \Delta')$ and $h'(p') = t$ for h' in S_h. We can in the same sense represent

M as a kind of generalized "random variable" over $\boldsymbol{H}$, for M also corresponds with a mapping which associates with each h in H, the Borel set Δ of "allowable M-values," i.e., M takes h to Δ where $p = (M, \Delta)$ is the "least M-predicate" defined above.

3. *The strongest statement that can be made in state ϕ is that the measurement of A yielding a_k is $|(\alpha_k, \phi)|^2$*: It follows from the logical analysis that in quantum theories, where maximal valuations are not bivalent, only statistical assertions are possible from ϕ about the outcome of a measurement, since in general the predicate (A, a_k) is not decided in ϕ. If it happens to be the case that $h(A, a_k) = t$, then in this case $\text{prob}_\phi((A, a_i){:}A) = 1$, as we have seen in Lemma 2.6. But since in a quantum theory even if ϕ is maximal it is not a bivalent valuation, then for some other magnitude B it will be the case that $S_\phi^{(B,bk)} \neq S_\phi^B$, so that in this case $\text{prob}_\phi((B, b_k){:}B) \neq 1$. Thus state ϕ will generally make only statistical assertions about values of the other magnitudes. The logical analysis agrees that quantum theories, which do not have bivalent states, will be irreducibly statistical.

4. *Measurement causes drastic changes in the state of a system:* In quantum mechanics we know that regardless of the state vector before measurement, afterward it will coincide with the eigenvector corresponding to the eigenvalue obtained in the measurement. This coincides exactly with the logical analysis, since a new state after measurement must contain the valuation h_M, characteristic of the outcome of the measurement M. Furthermore, as we saw in Lemma 2.3, the change from h to h_M in quantum theories will generally involve a drastic alteration of state. In classical theories there may be a change in valuation, but not of state.

5. *There are incompatible observables, and uncertainty relations:* Where maximal valuations are not bivalent there must be magnitudes M, N such that no valuation h decides all M- and all N-elementary predicates (for otherwise the maximal valuations would be bivalent). In these cases the order of magnitudes measured will determine probabilities in a sequence (shown in Lemma 2.5). Furthermore, where states are not bivalent, there will also be uncertainty relations, as one can easily show. If we define *incompatible magnitudes M, N* as magnitudes whose predicates are not decided together in any valuation, then we can derive the logical analogue of the uncertainty relations:

Lemma 2.7. If M, N are incompatible, p any M-predicate, q any N-predicate, then as $\text{prob}_h(p) \to 1$, $\text{prob}_h(q) \to 0$ any h in H.

As the probability according to h of an M-predicate p approaches 1, the probability according to h of an incompatible N-predicate q approaches 0.

For as $\text{prob}_h(p) \rightarrow 1$, $S_h^p \rightarrow S_h$ by the fundamental definition and since M, N are incompatible there is no valuation which finds both p and q true, and so as $S_h^p \rightarrow S_h$, $S_h^q \rightarrow \varnothing$, and hence $\text{prob}_h(q) \rightarrow 0$.

We see from this discussion that if the logical analysis is accepted, all the supposedly intractable peculiarities of quantum probabilities can be explained. For the first time we have a framework for understanding the quantum probabilities as probability measures over a probability space in a way that gives clear meaning to the Hilbert space representation. This analysis rests on clear logical foundations which take the strong conditional (i.e., the relative or transitional probabilities of quantum mechanics) as the fundamental probabilities, showing how the classical probabilities arise as a special case.

We therefore agree for example with a recent author that "the novelty [of quantum mechanics] is more related to the broadening [of probability] theory than to its falsification" (Constantini, 1993). We agree with this author, too, on the need for "a more accurate study" of the foundations of quantum probabilities than is usually given, a need he clearly articulates and supports with specific examples in his paper.

Furthermore, there are no outrageous philosophical assumptions or implications of this view. Here we made no outside appeal to metaphysics, and required no drastic changes to accepted analysis of logical operations or of probability theory. Instead we have simply proposed that the probabilities of any mechanical theory are measures over a field of sets which is not generated from the entire set of states in the system, but instead from a subset dependent on the initial condition. In classical theories we showed this can always be reduced to a "weak" conditional on the full state space, but in quantum theories such a reduction is not possible. This strongly conditional nature of quantum probability explains the peculiarities of quantum transitional probabilities. We have seen that the Hilbert space representation can be naturally viewed as a representation of these strongly conditional probabilities.

The logical analysis thus has major implications for our understanding of quantum mechanics, some of which will be explored in the next section. We see, for example, that on this view the logic used by all theories is the same, and like the valuations and states of any theory, this logic has the underlying structure of a distributive, though only relatively orthocomplemented lattice. Furthermore, we have seen that it is solely a difference in the structure of their maximal valuations, specifically the lack of bivalent states, which leads to the use of strongly conditional probabilities in quantum theories and hence generates all their nonclassical peculiarities.

3. REVIEW

3.1. Distributivity and "Quantum Logic"

One important consequence of the present view is that the propositional logic used by classical and quantum theories is essentially the same, with the structure of a distributive though only relatively orthocomplemented lattice. Furthermore, the structure of valuations and hence of quantum states is also distributive and relatively orthocomplemented. It follows that according to this analysis the Hilbert space representation of quantum states does *not*, as is widely believed, impose a nondistributive structure on quantum theories. This is contrary to present accepted opinion, and so the arguments which supposedly support nondistributivity will now be closely examined.

The present wide support for the view that quantum theories are nondistributive is surprising since distribution is a fundamental property of deduction and thus of science itself. It is also a key mathematical property of set inclusion and although nondistributive "orthomodular" lattices have been widely studied, they have none of the mathematical simplicity or significance of the distributive structures. Furthermore, nondistributivity poses great problems for the interpretation of quantum mechanics since it leaves us in doubt about our own ability to reason about, or draw deductions from, quantum theories. And lastly support for this view is surprising given how little mathematical effort has been expended on *proving* nondistributivity of quantum structures, especially when compared with the really huge effort which has now been expended investigating nondistributive alternatives. In this section we first dispatch what logical argument there is to support a nondistributive "quantum logic," then try to reconstruct the real basis for the nondistributive view, which apparently lies in the Hilbert space representation of quantum "properties."

The argument that quantum logic is nondistributive was first proposed in a joint paper by Birkhoff and von Neumann, entitled "The Logic of Quantum Mechanics" (Birkhoff and von Neumann, 1936). After some discussion of the issues the authors purport to show that distribution fails by considering a single experimental situation which they describe as follows:

> That [distribution] does break down is shown by the fact that if **a** denotes the experimental observation of a wave packet φ on one side of a plane in ordinary space, **a**′ correspondingly the observation of φ on the other side, and **b** the observation of φ in a state symmetric about the plane, then (as one can readily check) $b \cap (a \cup a') = b \cap ■ = b > © = (b \cap a) = (b \cap a') = (b \cap a) \cup (b \cap a')$. (Birkhoff and von Neumann, 1936, p. 10)

Symbols ■ and © represent the logically true and logically false proposition, respectively. A detailed treatment of this argument is given in Garden (1984, pp. 143ff). Here we simply note that this logical argument clearly relies on the assumption that $(a \cup a') = ■$, i.e., that this disjunction is logically true. According to our own analysis this assumption is not accepted and so the argument that distribution fails breaks down.

Almost every discussion of the "failure" of distribution in quantum logic relies on considering this special case where two complements are combined, a fact which should at least raise suspicions that it is the properties of negation, not implication, which are at issue. Once lattice complementation or Excluded Middle are assumed, it is easy to deduce the failure of distribution as shown above. But in all such examples one can just as well argue that it is a lack of full lattice complement, i.e., a weaker negation, and not nondistributive implication, which causes the Boolean identity to fail. In fact since distribution is fundamental to a partial ordering and hence more important to logic by far than a strong negation operator, we should more properly use this argument as a *reductio* proof that Excluded Middle fails!

Birkhoff and von Neumann's argument is specifically presented in logical terms, and it is analyzed in these terms above—negation does not obey Excluded Middle, thus $(a \cup a') \neq \blacksquare$ and the failure of distribution cannot be derived. However, we often see similar arguments which appeal explicitly to set-theoretic rather than logical principles, appealing in particular to supposed properties of set combination in Hilbert space. Consider, for example, the following argument from Jauch, typical both for its brevity and also for the use of one single case involving complements to establish the "failure" of distribution:

> Let us examine a very special case which displays the characteristic features of the general situation. We take a two-dimensional Hilbert Space H and choose two one-dimensional subspaces M and $M^{\perp}$ for instance. Let N be any one-dimensional subspace $\neq M$, $\neq M^{\perp}$; then we have $N \cap (M \cup M^{\perp}) = N \cap H = N$, but $N \cap M = \varnothing = N \cap M^{\perp}$. We see therefore that the operations $\cup$ and $\cap$ do not always satisfy the distributive law as they do in the case for sets. (Jauch, 1968, p. 27)

If this were expressed in logical terms, it would be exactly analogous to the example of Birkhoff and von Neumann above, and one would simply point out here, as there, that $(M \cup M^{\perp}) = H$ is assumed, and that were this assumption to be dropped, distribution would not fail. However, if the argument is taken at face value, i.e., as an argument about combining subspaces, then it, too, relies on the unwarranted assumption that $(M \cup M^{\perp}) = \boldsymbol{H}$. For although M, $M^{\perp}$ are orthogonal in the Hilbert space, i.e., $(M \cap M^{\perp}) = \varnothing$, we know also that $N \in \boldsymbol{H}$, and yet $N \notin (M \cup M^{\perp})$ and so $(M \cup M^{\perp}) \neq \boldsymbol{H}$. Thus $N \cap (M \cup M^{\perp}) \neq N \cap H$, and distribution does not fail.

Elsewhere in his book Jauch does discuss the logic of mechanical theories and in fact adopts the following as a postulate or axiom: "The propositions of a physical system are a complete, orthocomplemented lattice" (Jauch, 1968, p. 77). This postulate is presented in spite of earlier discussion which includes the claim that "We introduce here a strong negation denoted by 'false' and distinguished from simply 'not true' " (*Ibid.*, p. 76). By explicitly adopting Excluded Middle, however, as a principle of negation [a few sentences previously (*Ibid.*, p. 76)] and then, as we see above, postulating that the logic is a fully complemented lattice, Jauch actually ensures that such a distinction cannot be made for negation in his system. And of course once Excluded Middle is accepted, and hence full lattice complementation postulated, the failure of distribution will inevitably follow by arguments similar to those above. Jauch's justification for this postulated negation seems to rely solely on the fact that a 0 and a 1 element exist in the lattice representing a logic, since the 0 represents a logically false and the 1 a logically true proposition, respectively. We also agree on this point (see Section 1.2 above), but this does not of course show that negation must be represented by full complementation *with respect to these* 0 *and* 1 *elements*. Instead it has been argued here that negation is represented by complementation relative to a subsystem of the lattice, i.e., relative to the 0 and 1 elements *of a wff's logical "context."*

To really understand why lattice complementation or equivalently the Excluded Middle is almost invariably assumed as an axiom in discussions of mechanics, one needs in fact to look beyond these logical or set-theoretic arguments to the representation of quantum theories in Hilbert space. In assuming $(M \cup M^{\perp}) = H$ in his example, Jauch is perhaps appealing to the fact that a measurement of M on a system initially described by ϕ must find the system in one of just two eigenstates of an associated operator P_M, i.e., $P_M\phi = \phi$ or $P_M\phi = 0$ (see section 2.4 above). It is this fact also perhaps which motivates Birkhoff and von Neumann's assumption that $(a \cup a')$ is a logical truth in quantum logic, i.e., that $(a \cup a') = \blacksquare$; see above. Yet careful accurate analysis does not support their assumptions. Contrary to almost universal opinion, a careful analysis of the Hilbert space representation of quantum properties does not establish that the propositions, properties, or subspaces are nondistributive.

To assess the arguments we need a careful examination of the relations among *observables, "properties," projections,* and *subspaces* of Hilbert space. There is no better nor more authoritative work on this subject than von Neumann's own great book (von Neumann, 1955).

Recall that any observable operator A can be decomposed according to the Spectral Theorem into associated subspaces—in the case of simple A these are the eigenvectors $\{\alpha_i\}$ of A and in the more general case this is a decomposition into analogous subspaces associated with intervals in the spectrum of A. The probabilities assigned by vector ϕ are generated by the "size" of the components of ϕ along these axes, or alternatively the "size" of the projection of ϕ into the subspaces (see Section 2.4 above). This decomposition allows von Neumann to associate with any observable A a set of *properties*, represented by the *projection operators* of the spectral decomposition of A. These are themselves "observable operators," as von Neumann points out:

Apart from the physical quantities **R** there exists another category of concepts that are important objects of physics, namely the properties of the states of the system S. Some such properties are: that a certain quantity **R** takes the value λ, or that the value of **R** is positive, or that the values of two simultaneously observable quantities **R**, **S**, are equal to λ and μ respectively, or that the sum of the squares of these values is > 1, etc. We denote the quantities by **R**, **S**, . . and the properties by **E**, **F**. . The hypermaximal Hermitian operators $R, S, \ldots$ correspond to the quantities as was discussed above. Now what corresponds to the properties?

To each property **E** we can assign a quantity which we define as follows: each measurement which distinguished between the presence or absence of **E** is considered a measurement of this quantity, such that its value is 1 if **E** is verified, 0 in the opposite case. This quantity which corresponds to **E** will also be denoted by **E**. (von Neumann, 1955, p. 249)

Note that von Neumann uses "quantity" where we have used "magnitude." He then proceeds to show the relation between the *projection operators* corresponding to these *properties*, and *subspaces* of the Hilbert space:

The projections E therefore correspond to the properties **E** (through the agency of the corresponding quantities **E** which we just defined). If we introduce, along with the projections E the closed linear manifolds belonging to them ($E = P_M$), then the closed linear manifolds correspond equally to the properties of **E**.

If in state ϕ we want to determine whether or not a property **E** is verified, then we must measure the quantity **E**, and ascertain whether its value is 1 or 0 (these processes are identical by definition). The probability of the former, i.e. that **E** is verified, is consequently equal to the expectation value of **E**, i.e. $(E\phi, \phi) = \|E\phi\|^2 = \|P_M\phi\|^2$ and that of the latter, i.e. that **E** is not verified, is equal to the expectation value of $1 - \mathbf{E}$, i.e. $(1 - E\phi, \phi) = \|(1 - E)\phi\|^2 = \|\phi - P_M\phi\|^2$. The sum is of course equal to (ϕ, ϕ), i.e. to 1. Consequently **E** is certainly present or certainly absent, if the second or first probability respectively is equal to zero, i.e. for $P_M\phi = \phi$ or $P_M\phi = 0$. That is, if ϕ belongs to **M** or is orthogonal to **M** respectively; or if ϕ belongs to **M** or to $\mathfrak{R}_\infty - \mathbf{M}$ (*Ibid.*, p. 250)

Von Neumann now proceeds to show how properties and the associated subspaces can be combined:

It is clear that **E**, **F** are simultaneously decidable if and only if the corresponding quantities **E**, **F** are simultaneously measurable (whether with arbitrarily great or with absolute accuracy is unimportant since they are capable of the values 0, 1 only), i.e. if E, F commute. The same holds for several properties **E**, **F**, **G** . . .

From properties **E**, **F**, which are simultaneously decidable we can form the additional properties "**E** and **F**" and "**E** or **F**". The quantity corresponding to "**E** and **F**" is 1 if those corresponding to **E** and to **F** are both 1 and it is 0 if one of these is 0. Hence it is the product of these quantities, . . . its operator is then the product of the operators of **E** and **F**, i.e. EF, . . and, the corresponding closed linear manifold is the set **L** common to both **M** and **N**. (*Ibid.*, p. 251)

A similar analysis shows that "**E** or **F**" is associated with projection operator $E + F - EF$, and hence with the linear manifold $\mathbf{M} + (\mathbf{N} - \mathbf{L})$. He finishes the discussion with this conclusion:

> As can be seen, the relation between the properties of a physical system on the one hand and the projections on the other, makes possible a sort of logical calculus with these. However in contrast to the concepts of ordinary logic, this system is extended by the concept of "simultaneous decidability" which is characteristic for quantum mechanics. (*Ibid.*, p. 253)

We see here that von Neumann's own conclusion does not support the view later argued in the joint paper with Birkhoff, that quantum theories are nondistributive. Instead he has concluded that complex properties are not always defined.

The analysis of properties can be summarized thus: According to von Neumann, we can consider for any *magnitude* **R** some particular value or range of values λ and this corresponds to a "*property*" **E** which is also a magnitude and can be associated with a *projection operator* E which is an observable operator and which characterizes a *subspace* **M**. We determine whether the property "obtains in a state ϕ" by measuring the magnitude **E**, then determining whether $E\phi = 1$ or $E\phi = 0$ after the measurement, or correspondingly whether $P_{\mathbf{M}}\phi = \phi$ or $P_{\mathbf{M}}\phi = 0$ corresponding to property **E** being certainly true after the measurement or $1 - \mathbf{E}$ being certainly true, respectively. This analysis does establish a natural relation among properties, subspaces, and projection operators.

However, in considering whether these properties or the associated subspaces are nondistributive we need to consider how they are *combined*, and this requires that we determine how complex properties are formed from simple ones. It follows from von Neumann's own characterization quoted above that properties are *predictions about the certainty of outcomes of measurement*. Von Neumann's properties (or Jauch's propositions or "yes/no experiments") are not therefore truth-functional propositions or predicates in any standard logical sense. "Connectives" among properties, unlike truth-functional logical connectives, will not always be defined. For two properties can only be combined to form a new property if their constituents are simultaneously decidable. Where constituents are not simultaneously decidable we cannot find a magnitude whose measurement has all constituents as outcomes, and so no complex property can be defined in terms of them, i.e., we cannot consider their "certainty after measurement." In classical theories since states are bivalent, all elementary propositions are simultaneously decidable in any classical state, so this is not an issue. But in quantum theories, as von Neumann remarks, properties will only be combined if associated magnitudes are compatible.

Once we appreciate the nature of properties in this sense, we see that arguments for nondistributivity in quantum theories arise from confusion about the combination of elementary into complex properties. It is generally accepted that where magnitudes are compatible corresponding set relations among subspaces of Hilbert space are also completely classical, i.e., the set combinations are distributive. Arguments for the failure of distribution rely on cases where the subspaces combined correspond to incompatible magnitudes. Yet we have seen that in such cases corresponding complex properties in fact do not exist, i.e., the combinations of constituent properties are not well defined, and thus questioning the distributivity of their combination is not an issue.

We attempt to make these points completely clear by expressing von Neumann's analysis in logical terms, using the modal extensions of L which were sketched in Sections 2.1 and 2.3 above. It is clear that the "properties" as defined by von Neumann do not correspond to predicates or simple truth-functional propositions. They are expressed instead by modalities since they correspond to "certainty of outcome after measurement." Thus the *property* that (M, Δ) is certain to be true after a measurement of M on initial state ϕ, corresponding, say, to $P_M\phi = \phi$ or $P_M\phi = 0$ for initial ϕ, does not simply represent the predicate $p = (M, \Delta)$, but instead the much stronger *modal expression* $\Box_M p$. Although predicates and properties correspond in a natural way, the structure of each and the rules for combining each will be entirely different. As we have seen, predicates take truth-values to express propositions, and their combination can be analyzed as a generalized classical propositional logic with distributive hook but generalized negation. Properties, however, are represented by modal wffs and are properly analyzed in a modal extension of this propositional logic.

Definition. In the modal extension of L, LM, we can say wff α is a *property* (an M-property) if $\alpha = \Box_M p$ for $p = (M, \Delta)$, $p \in L$.

Properties can be analyzed within LM. For example, one can show the following:

Lemma 3.1. Where $p = (M, \Delta_1)$, $q = (M, \Delta_2)$, p, $q \in L$, then

$$\models (\Box_M p \vee \Box_M q) \equiv \Box_M (p \vee q)$$

Thus the disjunction of two M-properties is here shown to be itself an M-property. Recall that $(p \vee q) = (M, \Delta_1 \cup \Delta_2)$ and so $\Box_M(p \vee q)$ is a property according to the definition. The equivalence in Lemma 3.1 follows simply from the definitions. A special case of this result has particular interest:

Lemma 3.2. For $p = (M, \Delta)$, $p \in L$, then in LM

$$\models(\Box_M p \vee \Box_M p^\perp) \equiv \Box_M(p \vee p^\perp)$$

After measurement of M some M-outcome must be true, thus the trivial M-predicate is certain, and so, too, is one of the two disjuncts p, $p^\perp$, by definition of the elementary predicates. This follows in a trivial way from Lemma 3.1, the discussion of measurement in Section 2.3, and the form of the elementary predicates. Lemma 3.2 thus provides us with what may be viewed as a "weak version" of the Excluded Middle, $\models\Box_M(p \vee p^\perp)$, which *does* hold for properties in any theory.

We can also express, using the modalities, one of the key differences between properties in a classical and a quantum theory. First we define a new modal operator to express "certainty in every state":

$$(\mathrm{V}\Box_S) \quad h(\Box_S\alpha) = t \text{ iff } h^*(\alpha) = t \ \forall h^*\text{: } h^* \text{ is a state of } T$$

If we extend LM to include this operator and valuation rule ($\mathrm{V}\Box_S$), we can make explicit a key difference between properties in classical and in quantum theories:

Lemma 3.3. For every $p = (M, \Delta)$, $p \in L$, then in the modal extension LM: *In a classical theory* $\models\Box_M(p \vee p^\perp) \equiv \Box_S(p \vee p^\perp)$. *In a quantum theory* this is not the case.

In classical theories the fact that $(p \vee p^\perp)$ is certain after measurement of M ensures that $(p \vee p^\perp)$ is true in every state, by the bivalence of states and Lemma 2.1. In quantum theories this does not hold. Although $(p \vee p^\perp)$ is certain to be true after measurement of M given any initial h, it is not certain to be true in every state.

Confusion between the two modalities expressed here by $\Box_M$ and $\Box_S$ most likely accounts for the nearly universal assumption that Excluded Middle is a logical truth of mechanical theories, or equivalently that the descriptions of mechanics must be represented by a lattice with full lattice complementation. It is the result in Lemma 3.2, the "weak version" of Excluded Middle, which maybe motivates Birkhoff and von Neumann to assume in their example that $(a \cup a') = \blacksquare$, and Jauch to assume that $(M \cup M^\perp) = H$. But this "weak Excluded Middle" does not justify their much stronger assumptions, as we see in Lemma 3.3. In a theory without bivalent states, certainty of outcome after measurement is not the same truth in all states, or as logical truth.

Another key difference between classical and quantum properties can be expressed in the modal language, namely the ways in which M- and N-properties are combined.

Lemma 3.4. For $p = (M, \Delta)$, $q = (M^*, \Delta^*)$, $p, q \in L$, $M \neq M^*$, then in extension LM: *In classical theory* $\models(\Box_M p \wedge \Box_{M^*} q) \equiv \Box_N(p \wedge q)$ some magnitude N of T. *In quantum theory* this is not the case.

In classical theories elementary properties can always be combined using logical connectives to form complex properties, even when the predicates contain distinct magnitudes. In quantum theories such combinations do not in general yield properties. Thus the M- and M^*-predicates can always be combined by logical connectives to form wffs (though these may not be assigned a truth-value and in this sense may not express a proposition), but their corresponding properties will not generally combine to form new properties, because of the incompatibility of magnitudes.

Finally, according to the logical analysis there is indeed a natural relation between "properties" in the modal logic and "subspaces" of valuations in the Hilbert space representing this theory. Each value or range of values Δ of magnitude M is associated with the characteristic valuation h_p where $p = (M, \Delta)$, and this valuation contains all the "atomic" valuations or rays h_{pi} such that $p_i = (M, r_i)$ for $r_i \in \Delta$. Furthermore, the certainty of p being true after a measurement of M according to h corresponds to the weak conditional probability

$$\mathrm{prob}_h(p{:}M) = \mu_h(S_h^{p\wedge M})|\mu_h(S_h^M) = \mu_h(S_h^p)|\mu_h(S_h^M)$$

being 1, and this is the case when $S_h^p = S_h^M$. When $S_h^p = S_h^M$, then p is certain to be found true after a measurement of M according to h, and hence we can say in this case that p is a certain outcome of M, or that the "property p" obtains. In this way property p according to initial h naturally corresponds to subspace $S_h^{p\wedge M}$, and in general property p can be seen to correspond to $S^{p\wedge M}$, i.e., to S^p.

It follows also from this analysis that where compatible magnitudes are concerned, subspace operations *do* indeed correspond to "connectives" among properties. We have already seen, for example, that

$$\mathrm{prob}_h((p \vee \neg p){:}M) = \mu_h(S_h^p \cup S_h^{\neg p})|\mu_h(S_h^M) = 1$$

and so in this sense the disjunction of properties in Lemma 3.2 does correspond to set union among associated subspaces. But once more we see that where more than one magnitude is involved, corresponding subspaces may not exist, and so corresponding set combinations may not be defined. Suppose we let $q = (M^*, \Delta^*)$, then set $\alpha = (q \wedge (p \vee \neg p))$. This wff can always be constructed in L, but it only represents a *property* if there is a corresponding $\Box_N\alpha$ where N is a magnitude whose measurement decides all constituent predicates—and this will be the case only if the corresponding magnitudes M, M^* are compatible. In general M and M^* are not compatible and there is no set $S_h^{(q\wedge(p\vee\neg p))\wedge N}$ in H precisely because there is no such N.

Finally we turn again to Jauch's example and consider it now as an argument about properties in von Neumann's sense. First we note that while $(M \cup M^{\perp})$ is certain to be true after measurement of M, and in this sense it does represent a complex property, $(M \cup M^{\perp})$ will not be true in every state (Lemma 3.3), and hence we do not accept that $(M \cup M^{\perp}) = H$. Furthermore, Jauch chooses N such that $N \cap M = \varnothing = N \cap M^{\perp}$, which is to say that these magnitudes M and N are not compatible. But it therefore follows that $N \cap (M \cup M^{\perp})$ fails to be a property and in particular $N \cap (M \cup M^{\perp}) \neq N$. Distributivity does not fail!

Arguments from properties or their associated subspaces in Hilbert space which supposedly show that quantum properties or quantum set operations are not distributive in fact show only what von Neumann claimed at the outset—that we have to take into account "simultaneous decidability" in defining complex properties. Such arguments establish only that in quantum theories not all "properties" can be combined, because constituent predicates are not "simultaneously decidable," i.e., because all magnitudes are not compatible. And this of course we already know, for it corresponds exactly to the fact that there are no bivalent valuations in quantum theories.

3.2. Bell's Inequality

An argument similar to those discussed above but with even more shocking conclusions is based on the work of Bell and received credibility recently when experiment verified the failure of Bell's inequality in quantum theories. The physicist Bernard d'Espagnat has championed this view, and gives it very detailed discussion in d'Espagnat (1979, 1989) (see also Bell, 1987). D'Espagnat (1979), entitled Quantum Theory and Reality, gives a taste of his drastic conclusions in the subtitle: *The doctrine that the world is made up of objects whose existence is independent of human consciousness turns out to be in conflict with quantum mechanics and with facts established by experiment.*

D'Espagnat argues step by step from the "fact" that quantum properties cannot be added or "counted up" in the usual classical way to conclude that at least one of three fundamental assumptions of science, *Realism, Induction,* or *Locality*, must fail. He admits that each is fundamental to our world view, and describes them as follows:

> One is realism, the doctrine that regularities in observed phenomena are caused by some physical reality whose existence is independent of human observers. The second premise holds that inductive inference is a valid mode of reasoning and can be applied freely so that legitimate conclusions can be drawn from consistent observations. The third premise is called Einstein separability or Einstein locality, and states that no influence of any kind can propagate faster than the speed of light. (D'Espagnat, 1979, p. 128)

D'Espagnat takes a simple example of Bell's inequality concerning a system with spin defined in three different directions A, B, C, each having a possible value of just up or down. He thus represents the elementary propositions as pairs, using $A+$, for example, to indicate "the value of spin in direction A is up." He uses symbol $N(A + B-)$ to indicate the *number* of individual particles which have $A+$ and $B-$, for example, and then argues from basic set theory, for example, that

$$N(A + B-) = N(A + B - C+) + N(A + B - C-)$$

and

$$N(A + B-) \leq N(A + C-) + N(B - C+)$$

to conclude that

$$N(A + B+) \leq N(A + C+) + N(B + C+)$$

which is an expression of Bell's inequality (D'Espagnat, 1987, p. 135). As d'Espagnat points out, this inequality, while seemingly deduced from elementary set theory, is in fact violated in quantum probabilities, a fact which has since been supported by experiment. From this come the dire conclusions that the underlying premises of our world view fail.

D'Espagnat does make this remark in his paper:

> Another area that might be scrutinised for unacknowledged assumptions is the proof of Bell's Inequality. Indeed it seems the proof does depend on the assumed validity of ordinary, two valued logic, where a proposition must be either true or false and a spin component must be either plus or minus. Some interpretations of quantum mechanics have introduced the idea of a many-valued logic, but these proposals have nothing to do with the reasoning applied in this proof. Indeed in the context of the proof it is difficult even to conceive of an alternative to two valued logic. Unless such a system is formulated it seems best to pass over the problem. (d'Espagnat, 1987, p. 138)

Of course he "passes over the problem" of losing bivalent logic only to *confront* the problem of losing a fundamental principle of his science, and eventually reality itself!

According to the logical analysis of quantum theory, the failure of Bell's inequality is not only *not* paradoxical, *it is expected.* Whenever maximal valuations are not bivalent, these inequalities fail. For their "proof" follows from erroneous assumptions analogous to those discussed in Section 2.1 above. For any $h \in H$, $\alpha, \beta \in L$:

Lemma 3.5. In classical theories

$$\text{prob}_h(\alpha) = \text{prob}_h(\alpha \wedge \beta) + \text{prob}_h(\alpha \wedge \neg\beta)$$

In quantum theories

$$\text{prob}_h(\alpha) \neq \text{prob}_h(\alpha \wedge \beta) + \text{prob}_h(\alpha \wedge \neg\beta)$$

The result follows from the fact that $(\beta \vee \neg\beta)$ is not a logical truth, and hence $S_h^\alpha \neq S_h^{\alpha\wedge(\beta\vee\neg\beta)}$, so $S_h^\alpha \neq S_h^{(\alpha\wedge\beta)} \cup S_h^{(\alpha\wedge\neg\beta)}$ in general. However, where all maximal valuations are bivalent, the corresponding identities do hold. It follows that in considering the probabilities of the A, B, and C propositions in d'Espagnat's examples, it simply does not follow that $N(A + B-) = N(A + B - C+) + N(A + B - C-)$, for example, an assumption needed to derive the inequality. The lack of bivalence, as d'Espagnat himself notes, means that Bell's inequality fails.

Yet there is no mystery about the "many-valued" logic or about the "failure of classical bivalent logic." There are only two truth-values t, f, and the failure of bivalence simply corresponds to the use of partial valuations, not to the existence of some mysterious "extra" value. We are already familiar with the use of partial valuations in classical theories, for these correspond to probability assignments. Similarly the failure of bivalent logic is not mysterious. It was argued above that even in ordinary language we distinguish two senses of "not," so the generalisation to a nonbivalent logic brings us closer to ordinary discourse, not further from it. Algebraic analysis has shown that the nonbivalent system L is not a radical departure, it is distributive and relatively orthocomplemented, compared to the distributive, fully orthocomplemented structure of a Boolean algebra.

In fact, the failure of Bell's inequality can be reexpressed in terms of the failure of complex properties discussed above. We see in this example that we can introduce the property "spin in direction A," for example, with just the two values $+$ and $-$, and similarly we can introduce analogous properties in the B and C directions. We can relate these properties to specific measurements so that we know exactly in which circumstances $A+$ and in which cases $A-$, for example, can be said to hold. But as von Neumann pointed out in his original analysis, such properties in quantum theories cannot always be *combined.* Since the A, B, and C propositions are not all simultaneously decidable, i.e., since there is no measurement D which can have each of these as outcomes, *there is no complex property* of the form $(A+, B-, C+)$, for example. And since there are no such complex properties, it is hardly surprising that Bell's inequality, and other expressions which are derived by assuming they are well defined, fail in quantum theories.

It follows that we must be more careful in quantum theories than in classical ones to distinguish properties as theoretical terms from real features of the system described. Paradox results if we talk of "real properties" not combining in an expected way, while no paradox arises if we admit that our theoretical terms, our "predictions about measurement" cannot be meaningfully combined. In quantum theories it appears that our descriptions may not "perfectly fit" the reality they describe, in the sense that maximal valuations are not bivalent, so more than one state may be needed to describe the same reality, and simple properties in the theory cannot always be combined into complex properties.

So according to the logical analysis of Bell's inequality we merely point out that since maximal valuations of quantum theory are not bivalent and therefore magnitudes are not compatible, the disjunction $A+ \vee A-$ is not a logical truth. It is a dichotomy imposed by our choice of magnitudes and the structure of mechanical predicates, but it is not a logical truth. Because there are no bivalent valuations, there will be some states unable to find either of the disjuncts true. We know that no quantum state can consistently combine truth-assignments to A, B, and C propositions, and so we know that the corresponding Bell inequalities fail. Yet this failure does not indicate a breakdown in the set-theoretic structure of either the logic, the properties, or the probability space used in quantum theories. It simply demonstrates that the magnitudes used in quantum theories to describe reality are not always compatible or equivalently that these theories do not have bivalent states. Experimental support of the failure of Bell's inequalities merely confirms this fact.

3.3. Understanding Quantum Mechanics

We have seen that according to this analysis all the quantum peculiarities are explained by the lack of bivalent states in quantum theories. The logical laws are the same in classical and quantum mechanics, and set operations, underlying axioms of science, and indeed our comprehension of reality itself do not break down. Instead the fact that quantum theories use incompatible magnitudes whose elementary predicates cannot all be consistently combined in a single state-description is sufficient to explain the Hilbert space representation and to understand the quantum transitional probabilities as probabilities in a well-defined sense. We have seen that where consistent bivalent valuations cannot be made from the elementary predicates, probabilities become strongly conditional and the quantum peculiarities arise.

So at last we can see quantum mechanics not as a radical departure from classical concepts, but simply as a more general kind of theory using correspondingly generalized probabilities. In quantum theories we use strongly conditional probabilities, while in classical theories weak conditionals will always suffice. We see at last that both theories are legitimate descriptions of reality, neither departs radically from what we mean by a probability, nor do we have to reject fundamental principles of science, nor resort to drastic metaphysical assumptions inappropriate to the advancement of science. Instead we explain the quantum peculiarities simply as arising from one simple fact about our quantum descriptions—they cannot all be combined into consistent bivalent states.

In "interpreting" quantum mechanics we are therefore drawn to the question of why it is that these theories do use elementary predicates which cannot all be consistently combined into bivalent state descriptions, or alternatively why it is that such theories use incompatible magnitudes. Clearly there can be many different explanations or "interpretations" of this, all of which might be consistent with the logical analysis presented here. For example, it might be argued that it is the nature of subatomic reality itself which imposes incompatible magnitudes on quantum theory, or perhaps it is our own human brain which limits us to using these descriptions. Such views, being rooted in arguments specifically outside the logical considerations, will not be disproved by them.

However, the very independence of this logical analysis from physical or metaphysical assumptions makes such views implausible. Why appeal to outside considerations to explain features arising so obviously from the structure of the theory itself? We know from the analysis above that our choice of magnitudes and value-sets induces relations among elementary propositions and it is these relations which make theories incapable of bivalent states.

Of course one might conclude that such unique dispersion-free state descriptions are impossible, and after all this may prove to be the case. But why should we *assume* that this is so? In advancing science we should search for better descriptions of reality, and to do this we should not conclude *a priori* that improvements cannot be made.

There is after all nothing *self-evident* in our choice of magnitudes. The primitive magnitudes in quantum theory bring with them strong assumptions about the nature of reality which derive from classical mechanics and which may simply be inappropriate at the subatomic scale. Other descriptions might do the job better. For instance, the value-sets of mechanical magnitudes assume we are describing a "tiny billiard ball" which has position, for example, at one and only one point in space. When predicate (M, r_i) is true, for M a position magnitude, then predicates (M, r_j), r_j in V_M, must all be false by the very nature of the mechanical description, for $i \neq j$. Yet there are clearly other ways to describe reality which might involve quite different structures of elementary descriptions. For instance, we might assume matter *interacts* with space in the sense of creating a distortion in its spatial neighborhood, in which case descriptions in terms of a single mechanical "point position" predicate might be inappropriate. Or we might use a "field theory" such as Einstein proposed, where matter is regarded as a singularity of space-time itself, and this might require a different structure of elementary descriptions again. Clearly such alternative theories could still yield classical mechanics in the larger scale where, for example, localized distortions could be ignored. But since such a new theory would have elementary predicates and relations different from those used in quantum mechanics, it is at least conceivable that such a theory might also have bivalent states.

Perhaps this is the lesson of this logical analysis: we are describing reality using magnitudes that are ill-suited to the task. Although our descriptions are meaningful and useful and are confirmed by experiment, they are inadequate in a well-defined sense. Because the elementary predicates cannot all be consistently combined together, they leave us using only partial descriptions even for the states of a system, thus we need several states to describe the same reality, strongly conditional probabilities, measurement-dependent state descriptions, and irreducibly statistical states. All these peculiarities are a consequence ultimately of our choice of elementary description. Yet there may be other ways to conceptualize reality based on different magnitudes and value-sets, or indeed on a different form of elementary predicate. And it is quite conceivable, according to the logical analysis, that a theory with a different elementary structure may have bivalent states. In this case, as we have seen, it would have no incompatible magnitudes, no uncertainty relations, only weakly conditional probabilities, predictions which do not depend on measurement, and would have dispersion-free states uniquely describing individual realities.

It is perhaps interesting to turn finally to the debate between Bohr and Einstein of nearly 60 years ago in the light of this analysis. The present interpretation of quantum theory agrees with Bohr to the extent that we find language at the heart of the issue. It is our choice of magnitudes to describe reality which leads to the quantum effects. The use of magnitudes from classical mechanics clearly imposes a structure on elementary propositions which makes them incapable of all being consistently assigned truth-values, hence makes our theory nonclassical. To this extent we agree with Bohr. But Bohr added to his insight about the limitations of language a host of philosophical baggage which is not accepted here. For example, he maintained that we are necessarily bound by our classical concepts and can never break free from them, that subatomic reality is in principle indescribable, and that all subatomic theories will share the quantum peculiarities. These opinions are strongly rejected here. They seem founded purely on metaphysical prejudices completely inappropriate to the practice and advance of science!

Perhaps there is more common ground between the logical analysis and Einstein's view, pitted so passionately against Bohr's. Einstein always maintained there is an "incompleteness" in quantum theory, and that the irreducibly statistical nature of quantum states arises from a defect of the theory in the sense that some other theory might not have this feature and would thereby describe reality better. And with this we also agree. It is quite conceivable that another theory using different magnitudes could describe the same reality in a way which allows bivalent maximal valuations of its elementary propositions, and hence uses a single probability space, only weakly conditional probabilities, and dispersion-free states. Such a description, assuming it agreed with measurement, would describe reality better than quantum theories.

Yet it must be said, too, that Einstein's rebuttals of Bohr were grounded in a simplistic metaphysics which is not accepted here. He assumed that "every element of reality must have a counterpart in the physical theory" and attempted to define theoretical terms into ontological reality: "If without in any way disturbing a system we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity" (Einstein *et al.*, 1935). This is particularly ill-considered in the light of the analysis of properties above. But perhaps this naive metaphysics is not really the heart of his analysis. He considered that the quantum peculiarities arose from weaknesses in the theory, not from "God playing dice," and with that the present analysis agrees.

If, as it seems, we cannot add "hidden magnitudes" to generate bivalent states in existing quantum theories, then we should accept that the language of our present theory is flawed. The descriptions of quantum theory are "complete" in the sense of being maximal valuations, as "full" of elementary descriptions as possible. But they are not perhaps the best possible descriptions, not the "best fit" with reality in the sense that they do not allow bivalent states. Of course we cannot be sure we will ever find a "better fit," a theory with bivalent valuations, and hence dispersion-free states. But likely theories should be investigated on the grounds that they might provide a better description of reality.

So this interpretation at least reassures us that no fundamental law of logic has broken down, nor has set theory or reality deserted us. Instead we accept quantum theory as our present best description of reality and look forward to further developments. Like Einstein, we see no reason to suppose there is not some modified way of describing subatomic systems which agrees with experiment and yet decides all its propositions and thus yields dispersion-free states. And like Einstein, if there were such a theory, we would welcome it as an advance on quantum mechanics.

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